Random walk conditioned to stay positive*

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1 Introduction

A random walk that is certain to visit $(0, \infty)$ has associated with it, via a suitable h-transform, a Markov chain called random walk conditioned to stay positive, which will be defined properly in a moment. In continuous time, if the random walk is replaced by Brownian motion then the analogous associated process is Bessel-3. Let $\phi(x) = \log\log x$. The main result obtained here, which is stated formally in Theorem 1, is that, when the random walk has zero mean and finite variance, the total time that the random walk conditioned to stay positive is below x ultimately lies between $Lx^2/\phi(x)$ and $Ux^2\phi(x)$, for suitable (non-random) positive L and finite U, as x goes to infinity. For Bessel-3, the best L and U are identified.

Let $S = \{S_n\}$ be a random walk with independent identically distributed increments, with $S_0 = 0$ and with both $S_1 > 0$ and $S_1 < 0$ having positive probability. Let τ be the first time the random walk hits $(0, \infty)$, that is, the time of the first strict ascending ladder height. Assume that τ is proper. For $x > 0$, let $V(x)$ be the expected number of visits S_n makes to $(-x, 0]$ before first hitting $(0, \infty)$, so that

$$V(x) = E\sum_{j=0}^{\tau-1} I(S_j > -x),$$

and let $V(0) = 1$. It is fairly easy to check, and is Lemma 1 of [18], that, for any $s \geq 0$,

$$EI(S_1 + s > 0)V(S_1 + s) = V(s),$$

which implies that

$$I(S_i + s > 0, i = 1, 2, \ldots, n)V(S_n + s)$$

is a martingale. The corresponding h-transform produces the Markov chain, $\zeta = \{\zeta_n\}$, with

$$P((\zeta_1, \ldots, \zeta_n) \in A | \zeta_0 = s) = E\left[\frac{V(S_n + s)}{V(s)} I((S_1 + s, \ldots, S_n + s) \in A)\right]$$

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for $A \subset (0,\infty)^n$. Similar observations can be found in [4]. The Markov chain ζ is called random walk conditioned to stay positive for reasons given in [2]. Tanaka's main result in [18] is a pathwise construction of this chain (starting from zero) as a series of excursions between its successive future-minima, each of which is the minimum of the trajectory from that point on. Tanaka's construction, which is described more fully in the next section, will be the key to proving the following theorem concerning the time ζ spends below x as x goes to infinity.

Theorem 1 *Assume $ES_1 = 0$ and $E(S_1)^2 = 1$. Let $D(x) = \sum_n I(\zeta_n < x)$ and let $\phi(x) = \log\log x$ for $x > 3$. For suitable (non-random) L and U*

$$\limsup_{x\to\infty} \frac{D(x)}{x^2\phi(x)} \le U < \infty \quad \textit{and} \quad \liminf_{x\to\infty} \frac{D(x)}{x^2/\phi(x)} \ge L > 0$$

almost surely.

The progress of ζ to infinity could be examined in many ways. However, knowledge of the behaviour of $D(x)$ is needed in the study of a branching random walk with a barrier, discussed in [5], and, as a consequence of that study, also for results about certain functional equations, described in [6]. That application motivates focussing on $D(x)$. Furthermore, in that context, the particular values of L and U are of no consequence; nonetheless, some numerical information on L and U is recorded in the proof. It is also natural to ask whether in Theorem 1 the lim sup is strictly positive and the lim inf finite, thereby demonstrating that the norming functions are the right ones. This question is not considered, but analogous continuous time results, described soon, indicate that this will be so. Another way to examine the progress of ζ to infinity is to seek a 'minimal' envelope that ultimately contains the sample paths, in the style of the law of the iterated logarithm and its refinements. Recently, good results have been obtained on that problem in [12].

The assumption in Theorem 1 that S_1 has zero mean and unit variance means that the walk S oscillates. When the walk does not oscillate, $S_n \to \infty$ and then Tanaka's construction corresponds to the law of the random walk from its all-time infimum; see [2] and [9]. Hence, in those cases, which are not covered by Theorem 1 and not considered further, ζ goes to infinity in the same way as the random walk. Although it is not used here, it is worth pointing out that [2] also contains a different pathwise construction from Tanaka's.

As has already been mentioned, the simplest continuous analogue of ζ is the Bessel-3 process, denoted by R. When started from zero, this is a standard Brownian motion conditioned to stay positive. Most of the basic facts needed about R can be found in III.49 of [16]. Let $\mathcal{D}(x)$ be the time R spends below x. The following result is proved in the final section, but its proof is independent of the rest of the paper; its conclusions are stronger than those in the discrete case.

Theorem 2 *For a Bessel-3 process, R, let $\mathcal{D}(x) = \int I(R(t) < x)dt$ and let $\phi(x) = \log\log x$ for $x > 3$. Then*

$$\limsup_{x\to\infty} \frac{\mathcal{D}(x)}{x^2\phi(x)} = \frac{8}{\pi^2} \quad \textit{and} \quad \liminf_{x\to\infty} \frac{\mathcal{D}(x)}{x^2/\phi(x)} = \frac{1}{2}$$

almost surely and if $R(0) = 0$

$$\limsup_{x\downarrow 0} \frac{\mathcal{D}(x)}{x^2\phi(1/x)} = \frac{8}{\pi^2} \quad \textit{and} \quad \liminf_{x\downarrow 0} \frac{\mathcal{D}(x)}{x^2/\phi(1/x)} = \frac{1}{2}$$

almost surely.

The explicit constants here are compatible with the bounds obtained in the course of the proof of Theorem 1.

Most (if not all) of Theorem 2 is known, as are significant extensions. Note that $\mathcal{D}(x)$ is the time a 3-dimensional Brownian motion spends in a ball of radius x. Now see, in particular, Corollary 1.1 in [11] for the $\liminf$ part, which actually covers Bessel-d for $d \geq 3$ and Theorem 3 in [8] for the $\limsup$ part at 0, which also covers Bessel-d with $d \geq 3$ and integer. However, the Pitman representation of Bessel-3, introduced in [15], allows a proof to be given more easily for this case. Furthermore, aspects of the Pitman representation carry over to Lévy processes conditioned to stay positive, which is an obvious direction for generalization. It is also worth noting that $\mathcal{D}$ has the Ray-Knight representation $\mathcal{D}(x) = \int_0^x Y(y)^2 dy$, where Y is a Bessel-2, and so another direction for generalisation, suggested by Jon Warren, would be to consider $\mathcal{D}$ defined with Y taken as Bessel-d rather than Bessel-2.

The next section establishes some notation about ladder processes and describes Tanaka's construction of ζ. The following one contains a variety of preliminary estimates based on the construction. Sections 4 and 5 concern the upper and lower bounds, respectively, when the process starts from zero; together, these prove Theorem 1 in that case. In Section 6 the same estimates are shown to hold for any starting state, completing the proof of Theorem 1. Finally, the proof of Theorem 2 is discussed and some speculations are offered on its relationship with Theorem 1.

2 Ladder processes and Tanaka's construction

Recall that τ is the first time the random walk S hits $(0, \infty)$ and that τ is proper. The corresponding ladder height is S_τ. Denote the successive ladder times and heights by $\{(T_k^+, H_k^+) : k = 0, 1, 2, \ldots\}$, so that $\{(T_n^+ - T_{n-1}^+, H_n^+ - H_{n-1}^+) : n = 1, 2, 3, \ldots\}$ are independent variables, each distributed like $(T_1^+, H_1^+) = (\tau, S_\tau)$. Similarly, let $\{-H_k^- : k = 0, 1, 2, \ldots\}$ be the (possibly terminating) weak descending ladder height process of S. Denote the renewal measures corresponding to $\{H_n^-\}$ and $\{H_n^+\}$ by U^- and U^+, respectively. As usual, the renewal functions (the measure of $[0, x]$) are denoted by $U^-(x)$ and $U^+(x)$.

For a fixed j and $i \leq j$, let $S_i^* = S_j - S_{j-i}$; then, for a positive function f,

$$f(-S_j)I(\tau > j) \quad = \quad f(-S_j^*)I((j, S_j^*) \text{ is a descending ladder point}).$$

Taking expectations and then summing over j shows that

$$E\sum_{j=0}^{\tau-1} f(-S_j) = E\sum_{n=0}^{\infty} f(H_n^-) = \int f(y)U^-(dy).$$

In particular,

$$V(x) = E\sum_{j=0}^{\tau-1} I(S_j > -x) = \int I(y < x)U^-(dy).$$

Hence V agrees with the renewal function of U^- at points of continuity.

Let $\{w_i : i\}$ be independent copies of the segment of the walk up to τ viewed from (τ, S_τ) in reversed time and reflected in the x-axis, that is $\{w_i : i\}$ are independent copies of

$$(0, S_\tau - S_{\tau-1}, S_\tau - S_{\tau-2}, \ldots, S_\tau - S_1, S_\tau).$$

Now write $w_i = (w_i(j) : j = 0, 1, 2, \ldots, \tau_i)$ to identify the components of w_i. Tanaka shows in [18] that the Markov chain ζ can be constructed by gluing the w_i together, each starting from the end of the previous one, to give the trajectory, starting from $w_1(0) = 0$. More formally, let $\{(T_k^+, H_k^+)\}$ be the bivariate renewal process formed from the independent variables $\{(\tau_i, w_i(\tau_i))\}$, which has the same distribution as the bivariate renewal process formed by the strict ascending ladder times and heights of S and so is only a minor abuse of the notation already introduced. Then the Markov chain ζ is given by

$$\zeta_n = H_k^+ + w_{k+1}(n - T_k^+) \ \text{ for } T_k^+ < n \le T_{k+1}^+.$$

This construction gives the trajectory starting from zero (to which the process never returns) and shows that, in this case, using the law of large numbers, $\zeta_n \to \infty$ as $n \to \infty$. In Section 6, it is shown that the asymptotics considered are the same for any starting state; hence attention can centre on the process started from zero.

It is observed in [12] that the successive w_i can be taken from the path of S between its successive ascending ladder points, thereby removing the notational abuse of $\{(T_k^+, H_k^+)\}$. However, this pretty coupling of the trajectories of ζ and S, which is best grasped with a picture, is not important for the treatment here.

3 Preliminary estimates

From now on, assume that S_1 has a mean of zero and a variance of one. Assuming the mean is zero implies that both the strict increasing and weak decreasing ladder processes of the random walk are proper renewal processes. Then, the assumption that the variance is one implies that the mean ladder heights, b^+ and b^-, are both finite and that $2b^+b^- = 1$; see [10], especially Theorem 1 in XVIII.5.

The objective is to consider the asymptotic behaviour of the monotonic function $D(x)$. In the first instance, consider

$$D_k = \sum_n I(\zeta_n < H_k^+) = D(H_k^+);$$

the construction of ζ implies that only the first k of the w_i contribute to D_k. The next lemma gives a simple but important representation of D_k.

Lemma 1 *Let*

$$v_r(x) = \sum_{j=1}^{\tau_r} I(w_r(j) < x + w_r(\tau_r)).$$

Then

$$D_k = \sum_{r=1}^{k} v_r(H_k^+ - H_r^+)$$

and $(v_1, \ldots, v_r)$ *is independent of* $(H_k^+ - H_{k-1}^+, \ldots, H_k^+ - H_r^+)$. *Also,* $E[v_r(x)] = V(x)$ *and*

$$k^{-2} E D_k \to \frac{b^+}{2b^-} \text{ as } k \to \infty.$$

Proof. Splitting up the trajectory of ζ and then using $H_{r-1}^+ + w_r(\tau_r) = H_r^+$,

$$\begin{aligned} D_k = \sum_n I(\zeta_n < H_k^+) &= \sum_{r=1}^{k} \sum_{i=1}^{\tau_r} I(H_{r-1}^+ + w_r(i) < H_k^+) \\ &= \sum_{r=1}^{k} \sum_{i=1}^{\tau_r} I(w_r(i) < H_k^+ - H_r^+ + w_r(\tau_r)) \\ &= \sum_{r=1}^{k} v_r(H_k^+ - H_r^+). \end{aligned}$$

The independence claimed is immediate from Tanaka's construction.

Showing $E[v_r(x)] = V(x)$ is simply interpretation of the definitions of w_r and V. Then

$$E D_k = E \sum_{r=1}^{k} V(H_k^+ - H_r^+)$$

and dominated convergence and the renewal theorem give the limiting behaviour. $\square$

Some estimates about (a generic) v are needed to exploit this representation; these culminate in Lemma 4, which is geared to the production of (exponential) supermartingales in the next section. The main purpose of the next lemma is to bound the moments of v.

Lemma 2 *Let*

$$K(x) = E\left[(v(x) - V(x))^2\right]$$

and

$$W(x) = \sup\left\{ \int_0^y V(x - y + z) U^+(dz) : 0 \le y \le x \right\}.$$

(i) $K(x) = 2\int_0^x \int_0^y V(x-y+z)U^+(dz)U^-(dy) - V(x)^2 - V(x)$.

(ii) $x^{-3}K(x) \to 2/(3b^+(b^-)^2)$ *as* $x \to \infty$ *and so, for a finite* C, $K(x) \le C(x+1)^3$.

(iii) $E\left[v(x)^k\right] \le k! V(x) W(x)^{k-1}$.

Proof. For $x > 0$, let $V(x; y)$ be the expected number of hits of $\{S_n - y : n \ge 0\}$, which is the random walk started from $-y$, on $(-x, 0]$ before its first visit to $(0, \infty)$. Then $V(x; 0) = V(x)$. Now

$$E\left[\sum_{k=j}^{\tau-1} I(S_k > -x) \,\middle|\, S_0, S_1, \ldots, S_j \right] = I(\tau > j) V(x; -S_j).$$

Hence, using this for the third equality,

$$\begin{aligned}
K(x) + V(x)^2 &= E\left[\left(\sum_{j=0}^{\tau-1} I(S_j > -x)\right)^2\right] \\
&= E\left[2\sum_{j=0}^{\tau-1} I(S_j > -x) \sum_{k=j}^{\tau-1} I(S_k > -x) - \sum_{j=0}^{\tau-1} I(S_j > -x)\right] \\
&= 2E\left[\sum_{j=0}^{\tau-1} I(S_j > -x) V(x, -S_j)\right] - V(x) \\
&= \int_0^x 2V(x; y) U^-(dy) - V(x).
\end{aligned}$$

By considering the visits to $(-x, 0]$ between increasing ladder points of $S_n - y$,

$$V(x; y) = \int_0^y V(x - y + z) U^+(dz),$$

for $0 \le y < x$; see P19.3 in [17] and Theorem IV.20 in [3] for closely related results. Substituting this into the expression for $K(x) + V(x)^2$ proves (i) and estimation of the renewal functions yields (ii).

Let $I(k, n)$ be the collection of non-decreasing k-tuples from $\{1, 2, \ldots, n\}$. For a sequence $\{a_i : i = 1, 2, \ldots, n\}$ and $\alpha = (\alpha(1), \ldots, \alpha(k)) \in I(k, n)$, let a_α be the product $a_{\alpha(1)} a_{\alpha(2)} \ldots a_{\alpha(k)}$. Then, multiplying out,

$$\left(\sum_{i=1}^{n} a_i\right)^k \le k! \sum_{\alpha \in I(k,n)} a_\alpha.$$

Take $a_i = I(S_i > -x)$ and $n = \tau - 1$; then

$$(k!)^{-1} E[v(x)^k] = (k!)^{-1} E\left[\left(\sum_{i=1}^{n} a_i\right)^k\right] \le E\left[\sum_{\alpha \in I(k,n)} a_\alpha\right]$$

and, as already observed,

$$E\left[\sum_{i=j}^{n} a_i \,\middle|\, S_0, S_1, \ldots, S_j\right] = I(\tau > j) V(x; -S_j).$$

Hence, conditioning on the first $\alpha(k-1)$ steps for the second equality,

$$\begin{aligned}
E\left[\sum_{\alpha \in I(k,n)} a_\alpha\right] &= E\left[\sum_{\alpha \in I(k-1,n)} a_\alpha \left(\sum_{i=\alpha(k-1)}^{n} a_i\right)\right] \\
&= E\left[\sum_{\alpha \in I(k-1,n)} a_\alpha V(x; -S_{\alpha(k-1)})\right].
\end{aligned}$$

Now the bound $V(x; -S_{\alpha(k-1)}) \le W(x)$ and induction on k complete the proof. □

The function W, introduced in the previous lemma, played a significant part in the estimation there of the moments of v. The first part of the next lemma bounds W, thereby simplifying the estimates; the second is really only important for keeping the numerical value of U small in the next section.

Lemma 3 *(i) For some finite c, $(x+1)^{-2}W(x) \le c$ for all $x \ge 0$; (ii) $x^{-2}W(x) \to 1$ as $x \to \infty$.*

Proof. Since V and U^+ are increasing, $W(x) \le V(x)U^+(x)$ and the first part follows from bounding renewal functions. For the second part, note first that

$$\begin{aligned} x^2 W(x) &= x^2 \sup_{0\le y\le x} \int_0^y V(x-y+z)U^+(dz) \\ &\ge x^{-2}\int_0^x V(z)U^+(dz) \to \frac{1}{2b^+b^-} = 1 \end{aligned}$$

as $x \to \infty$, using renewal theory. Now fix $\epsilon > 0$. Then $x^{-2}W(x)$ can be bounded by the maximum of a finite number of terms of the form

$$x^{-2}\sup\left\{\int_0^y V(x-y+z)U^+(dz) : (\eta-\epsilon)x \le y \le \eta x\right\},$$

with suitable $\eta \in [0,1]$. Using monotonicity of V and then the renewal theorem,

$$\begin{aligned} x^{-2}\sup_{(\eta-\epsilon)x\le y\le \eta x}\int_0^y V(x-y+z)U^+(dz) &\le x^{-2}\int_0^{\eta x} V(x(1-\eta+\epsilon)+z)U^+(dz) \\ &\to 2\eta(1+\epsilon)-\eta^2 \end{aligned}$$

as $x \to \infty$. Since $2\eta(1+\epsilon)-\eta^2$ has a maximum of $1+2\epsilon$ for $\eta\in[0,1]$ it follows that $\limsup_{x\to\infty} x^{-2}W(x) \le 1+2\epsilon$. □

The proof of the upper bound relies on the following estimate of the moment generating function of $v(x)$, $Ee^{\theta v(x)}$.

Lemma 4 *For any $d < 1$ there is a κ, independent of x, such that*

$$E\exp(\theta(v(x)-V(x)) - \theta^2(K(x)+\kappa(x+1)^3)/2) \le 1,$$

for θ satisfying $0 \le \theta W(x) \le d$.

Proof. Take C such that $C > V(x)/(x+1)$ for all $x \ge 0$. Take C' such that, with c from Lemma 3(i), $d = C'/(C'+c)$; then $0 \le \theta W(x) \le d$ implies that

$$\frac{W(x)}{(x+1)^2}\frac{\theta W(x)}{(1-\theta W(x))} \le C'.$$

For these θ, using Lemma 2(iii),

$$\begin{aligned} \sum_{k=3}^{\infty}\frac{\theta^k}{k!}Ev(x)^k &\le V(x)\sum_{k=3}^{\infty}\theta^k W(x)^{k-1} \\ &\le \theta^2(x+1)^3\left(\frac{V(x)}{x+1}\right)\left(\frac{W(x)}{(x+1)^2}\frac{\theta W(x)}{(1-\theta W(x))}\right) \\ &\le CC'\theta^2(x+1)^3; \end{aligned}$$

now take $\kappa = 2CC'$. Then

$$\begin{aligned} E\exp(\theta v(x)) &\leq 1 + \theta V(x) + \frac{\theta^2(K(x)+V(x)^2)}{2} + \frac{\theta^2\kappa(x+1)^3}{2} \\ &\leq \exp(\theta V(x))\exp(\theta^2(K(x)+\kappa(x+1)^3)/2) \end{aligned}$$

for $0 \leq \theta W(x) \leq d$, as required. $\square$

The final preparatory lemma will be useful when an exponentially growing number of independent identically distributed variables has to be controlled with probability one.

Lemma 5 *Let $Y_{i,k}$ be identically distributed variables, with mean μ and $Y_{i,k}$ independent as i varies for fixed k. If $\liminf_k n(k+1)/n(k) > 1$ then, for any $\epsilon > 0$,*

$$\sum_k P\left(\left|\sum_{i=1}^{n(k)}(Y_{i,k}-\mu)\right| > n(k)\epsilon\right) < \infty.$$

Proof. See the proof of Theorem 10.2 in the Appendix of [1]. $\square$

4 The upper bound

The idea for obtaining the upper bound is the standard one of using exponential supermartingales. As already mentioned, Lemma 4 was designed to produce these.

Lemma 6 *If $\phi(x) = \log\log x$ (for $x \geq 3$) then, for a suitable (non-random) U,*

$$\limsup_{x\to\infty} \frac{D(x)}{x^2\phi(x)} \leq U < \infty.$$

Proof. Fix $d < 1$, choose κ as in Lemma 4 and, for notational convenience, let $u = 1/d > 1$. Let

$$\begin{aligned} X_s &= v_{n-s}(H_n^+ - H_{n-s}^+) - V(H_n^+ - H_{n-s}^+) \\ 2A_s &= K(H_n^+ - H_{n-s}^+) + \kappa(H_n^+ - H_{n-s}^+ + 1)^3. \end{aligned}$$

Note that

$$\sum_{s=0}^{n-1} X_s = D_n - \sum_{s=1}^{n} V(H_n^+ - H_s^+),$$

so information on $\sum_{s=0}^{n-1} X_s$ will translate into information on D_n.

For a fixed n, let $\mathcal{H}_n^r$ be the σ-field generated by $\{w_n, w_{n-1}, \ldots, w_{n-r+1}\}$. Then, given $\mathcal{H}_n^r$, $H_n^+ - H_{n-r}^+$ and $\sum_{s=0}^{r-1}(X_s - \theta A_s)$ are known; hence Lemma 4 and the fact that $W(H_n^+ - H_{n-r}^+)$ increases with r imply that for $0 \leq \theta y \leq d$

$$\left(I(W(H_n^+ - H_{n-r}^+) \leq y)\exp\left(\theta\sum_{s=0}^{r-1}(X_s - \theta A_s)\right), \mathcal{H}_n^r\right)$$

is a positive supermartingale.

Apply the maximal inequality, [14, II.2.7], to show that for $0 \le \theta y \le d$

$$P\left(W(H_n^+) \le y,\ \theta\sum_{s=0}^{n-1}(X_s - \theta A_s) > \theta a\right) \le e^{-\theta a}.$$

Now take $y = (nu^2b^+)^2$, $\theta y = d\ (= u^{-1})$, so that $\theta = 1/\gamma n^2$ where $\gamma = u^5(b^+)^2$, and $\theta a = u\phi(n)$ to give

$$P\left(n^{-2}\sum_{s=0}^{n-1} X_s - \gamma^{-1}n^{-4}\sum_{s=0}^{n-1} A_s > \gamma u\phi(n)\right) \le e^{-u\phi(n)} + P\left(W(H_n^+) > (nu^2b^+)^2\right).$$

Note that, for large $W(x)$, using Lemma 3(ii), $W(x)/x^2 \le u^2$ and so, for large n,

$$\begin{aligned} P\left(W(H_n^+) > (nu^2b^+)^2\right) &= P\left(\frac{W(H_n^+)}{(H_n^+)^2}(H_n^+)^2 > (nu^2b^+)^2\right) \\ &\le P\left(u^2(H_n^+)^2 > (nu^2b^+)^2\right) + P(H_n^+ < ndb^+) \\ &= P\left(H_n^+ > nub^+\right) + P(H_n^+ < ndb^+). \end{aligned}$$

Recall that H_n^+ is the sum of n independent copies of S_τ and $ES_\tau = b^+$.

Let $[x]$ be the integer part of x and let Σ be the sequence $\{[u^k]\}$, which should, more accurately, be written as $\Sigma(u)$. Then, by substituting for $\phi(n)$ and using Lemma 5,

$$\sum_{n\in\Sigma}\left(e^{-u\phi(n)} + P\left(H_n^+ > nub^+\right) + P(H_n^+ < ndb^+)\right) < \infty.$$

Hence

$$\limsup_{n\in\Sigma}\frac{\sum_{s=0}^{n-1} X_s}{n^2\phi(n)} \le \gamma^{-1}\limsup_{n\in\Sigma}\frac{\sum_{s=0}^{n-1} A_s}{n^4\phi(n)} + \gamma u.$$

Now, bounding crudely, using Lemma 2(ii) to produce a finite C_1,

$$\begin{aligned} 0 \le \frac{\sum_{s=0}^{n-1} 2A_s}{n^4\phi(n)} &= \frac{1}{\phi(n)}\frac{\sum_{s=0}^{n-1}(K(H_n^+ - H_{n-s}^+) + \kappa(H_n^+ - H_{n-s}^+ + 1)^3)}{n^4} \\ &\le \frac{C_1}{\phi(n)}\frac{n(H_n^+ + 1)^3}{n^4} \to 0. \end{aligned}$$

Similarly

$$\begin{aligned} 0 \le \frac{D_n - \sum_{s=0}^{n-1} X_s}{n^2\phi(n)} &= \frac{1}{\phi(n)}\frac{\sum_{s=1}^{n} V(H_n^+ - H_s^+)}{n^2} \\ &\le \frac{C_2}{\phi(n)}\frac{nH_n^+}{n^2} \to 0. \end{aligned}$$

Hence,

$$\limsup_{n\in\Sigma}\frac{D_n}{n^2\phi(n)} \le \gamma u = u^6(b^+)^2 < \infty,$$

and so, recalling the definition of D_n,

$$\limsup_{n\in\Sigma} \frac{D(H_n^+)}{(H_n^+)^2\phi(H_n^+)} = \limsup_{n\in\Sigma} \frac{D_n}{n^2\phi(n)} \frac{n^2\phi(n)}{(H_n^+)^2\phi(H_n^+)} \le u^6,$$

Finally, using monotonicity

$$\limsup_{x\to\infty} \frac{D(x)}{x\phi(x)} \;\le\; \limsup_{n\in\Sigma} \frac{D(H_n^+)}{(H_n^+)^2\phi(H_n^+)} u^2 = u^8.$$

This bound can be made arbitrarily close to 1. ☐

The numerical value of the bound goes back to the occurrence of $W(x)$ in the bound in Lemma 2(iii).

5 The lower bound

Standard estimates for independent identically distributed variables are used for the lower bound. This approach allows a straightforward treatment, but if the objective were to identify the best lower bound it throws rather a lot away.

Lemma 7 *Let Y_i be independent identically distributed non-negative variables with mean μ and variance σ^2. Let Z_n be the sum of the first n. Then, for any sequence $\{x_n > 0\}$ with, for some a, $x_n\mu \le a\sigma\sqrt{n}$ and for any $u > 1$ there is a (finite) $\lambda = \lambda(a, u)$ $(> \sqrt{2})$, which does not depend on the distribution of Y or on n, such that*

$$P(Z_n \le n\mu - \lambda\sigma\sqrt{n}x_n) \le \exp(-x_n^2 u).$$

By taking a and $(u-1)$ sufficiently small, λ can be taken arbitrarily close to $\sqrt{2}$.

Proof. The variables $\{\mu - Y_i\}$ are bounded above by μ. The result therefore follows from Lemma 10.2.1(i) of [7], with simplifications based on Corollary 10.2.1 there. ☐

Lemma 8 *If $\phi(x) = \log\log x$ (for $x \ge 3$) then, for suitable (non-random) L,*

$$\liminf_{x\to\infty} \frac{D(x)}{x^2/\phi(x)} \ge L > 0.$$

Proof. Informally, let $\psi(x) = \gamma^2/\phi(x)$ for some constant $\gamma > 0$. Formally, to make sure that $n\psi(n)$ is always an integer when n is, let $\psi(x) = x^{-1}[\gamma^2 x/\phi(x)]$; then $\psi(x)\phi(x) \to \gamma^2$ as $x \to \infty$. Now let

$$C_n = \sum_{r=1}^{n(1-\psi(n))} v_r\left(H_n^+ - H_{n(1-\psi(n))}^+\right);$$

then $D_n \ge C_n$. Let $\mathcal{G}^n$ be the σ-field generated by $\left\{w_n, \dots, w_{n(1-\psi(n))+1}\right\}$, so that, given $\mathcal{G}^n$, $H_n^+ - H_{n(1-\psi(n))}^+$ is known and

$$\left\{v_r\left(H_n^+ - H_{n(1-\psi(n))}^+\right) : r = 1, 2, \dots, n(1-\psi(n))\right\},$$

which are the variables occurring in C_n, are independent and identically distributed; denote their (conditional) common mean and standard deviation by μ_n and σ_n, respectively. Then, in terms of earlier notation,

$$\mu_n = V\left(H_n^+ - H_{n(1-\psi(n))}^+\right) \quad \text{and} \quad \sigma_n^2 = K\left(H_n^+ - H_{n(1-\psi(n))}^+\right).$$

The idea is to apply Lemma 7 to C_n, but three preliminary estimates are given first.

Let $u > 1$ and let Σ be the sequence $\{[u^k]\}$. Now, using the renewal theorem and Lemma 5,

$$\lim_{n\in\Sigma} \frac{\mu_n}{n\psi(n)} = \lim_{n\in\Sigma} \left(\frac{V\left(H_n^+ - H_{n(1-\psi(n))}^+\right)}{H_n^+ - H_{n(1-\psi(n))}^+} \right) \left(\frac{H_n^+ - H_{n(1-\psi(n))}^+}{n\psi(n)} \right) = \frac{b^+}{b^-},$$

almost surely. Similarly, using Lemma 2(ii) and Lemma 5,

$$\lim_{n\in\Sigma} \frac{\sigma_n}{(n\psi(n))^{3/2}} = \sqrt{\frac{2}{3}\frac{b^+}{b^-}}.$$

Thus, to control a in Lemma 7 when $x_n^2 = \phi(n)$,

$$\lim_{n\in\Sigma} \frac{\mu_n\sqrt{\phi(n)}}{\sigma_n\sqrt{n(1-\psi(n))}} = \lim_{n\in\Sigma} \frac{\gamma\mu_n}{\sigma_n\sqrt{n\psi(n)(1-\psi(n)}} = 0,$$

using the two limits just obtained.

Now apply Lemma 7 with $x_n^2 = \phi(n)$ to show that, for any $a > 0$,

$$P\left(C_n \le n(1-\psi(n))\mu_n - \lambda\sigma_n\sqrt{n(1-\psi(n))}\phi(n)^{1/2} \,\middle|\, \mathcal{G}^n\right)$$

is no greater than

$$I\left(\phi(n)^{1/2}\mu_n > a\sigma_n\sqrt{n(1-\psi(n))}\right) + \frac{1}{(\log n)^u} I\left(\phi(n)^{1/2}\mu_n \le a\sigma_n\sqrt{n(1-\psi(n))}\right).$$

Therefore, using conditional Borel-Cantelli and the estimates just obtained,

$$\begin{aligned}
\liminf_{n\in\Sigma} \frac{C_n}{n^2/\phi(n)} &\ge \gamma^2 \left(\liminf_{n\in\Sigma} \frac{(1-\psi(n))\mu_n}{n\psi(n)} - \lambda\gamma \limsup_{n\in\Sigma} \frac{\sqrt{(1-\psi(n))}\sigma_n}{(n\psi(n))^{3/2}} \right) \\
&= \gamma^2 \left(1 - \lambda\gamma\sqrt{\frac{2}{3}}\right) \frac{b^+}{b^-};
\end{aligned}$$

let the (finite, positive) maximum of this as γ varies be $\kappa(\lambda)b^+/b^-$. Then, using $D_n \ge C_n$, the definition of D_n and monotonicity,

$$\liminf_{x\to\infty} \frac{D(x)}{x^2/\phi(x)} \ge \kappa(\lambda)\frac{1}{b^+b^-} = 2\kappa(\lambda).$$

□

Since λ can be as low as $\sqrt{2}$, this bound can be made as big as 2/9; however, the whole calculation is based on the rather crude estimate $D_n \ge C_n$.

6 The starting state is irrelevant

The main result of this section implies that the assertion that $\zeta_n \to \infty$ as $n \to \infty$ and the almost sure asymptotic behaviours of $D(x)$ described in Lemmas 6 and 8 hold for any starting state, not just when starting from zero. The result is predictable and the ideas used its proof are standard, but it seems not to have been proved. Two lemmas are derived first. The first is similar to Lemma VII.12 in [3]; the second applies a standard conditioning argument.

Lemma 9 *Fix $x > 0$. Let $\mathcal{X}$ be the event $\{\zeta_n > x, n = 1, 2, 3, \ldots\}$ and, for $y \geq x$, let $Q_y = P(\mathcal{X}|\zeta_0 = y)$, which is the probability that, starting from y, the process never again drops as low as x. Then $Q_y = V(y - x)/V(y)$.*

Proof. Let Q_y^n be the probability the process does not fall as low as x in the first n steps. It will be convenient to let E_y be expectation for the random walk started from y, so that, in particular, for $y \geq x$

$$\begin{aligned} V(y-x) &= EI(S_i + y - x > 0, i = 1, 2, \ldots, n)V(S_n + y - x) \\ &= E_y I(S_i > x, i = 1, 2, \ldots, n)V(S_n - x). \end{aligned}$$

From the definition of ζ,

$$Q_y^n = E_y \left[I(S_i > x, i = 1, 2, \ldots, n) \frac{V(S_n)}{V(y)} \right].$$

Since V is, essentially, a renewal function there is a finite C such that $\sup_y |V(y) - V(y - x)| \leq Cx$. Therefore

$$\begin{aligned} |V(y)Q_y^n - V(y-x)| &= |E_y I(S_i > x, i = 1, 2, \ldots, n)(V(S_n) - V(S_n - x))| \\ &\leq CxE_y I(S_i > x, i = 1, 2, \ldots, n) \to 0 \end{aligned}$$

as $n \to \infty$. □

Lemma 10 *Tanaka's description applies to the height above x of the process ζ started from x and conditioned never again to fall as low as x.*

Proof. By Lemma 9, the conditioning event, $\mathcal{X}$, has positive probability from every state $y \geq x$. Direct calculation shows that the new process is a Markov chain, denoted by $\overline{\zeta}$, with transitions from $y \geq x$ having the law

$$P(\overline{\zeta_1} \in dz | \overline{\zeta_0} = y) = P(\zeta_1 \in dz | \zeta_0 = y) \frac{P(\mathcal{X}|\zeta_0 = z)I(z > x)}{P(\mathcal{X}|\zeta_0 = y)};$$

this law must now be shown to have the form claimed. Substituting for the transition law of ζ and using the notation of Lemma 9 give

$$P(\overline{\zeta}_1 \in x + A | \overline{\zeta}_0 = y) = E\left[\frac{V(y + S_1)}{V(y)} I(y + S_1 \in x + A) \frac{Q_{y+S_1}}{Q_y} \right]$$

for $A \subset (0, \infty)$. Applying Lemma 9 now shows that the transition law for $\left\{\overline{\zeta}_n - x\right\}$ is of the required form. □

Theorem 3 *When the process starts from $x > 0$, Tanaka's description applies to the development of the process from its all-time minimum.*

Proof. Each new 'weak minimum of the process up to now' defines a stopping time; let $T(i)$ be the ith of these. Each has a positive probability of being the all-time minimum; by Lemma 9, these probabilities are bounded below by $1/V(x)$ and so $I = \sup\{i : T(i) < \infty\}$ is finite and $T(I)$ is the time of the all-time minimum. Furthermore, by applying the strong Markov property and then Lemma 10, given $\{I = i\} = \{T(i) < \infty, T(i+1) = \infty\}$ the process from $T(i)$ onwards is as claimed. $\square$

This final argument is similar to that used in the proof of Theorem 2.3 in [2].

7 Continuous time

Proof of Theorem 2. The proofs for x going to infinity and to zero are very similar; the discussion focuses on the limit at infinity. Let R be a Bessel-3 process started from 0; Williams' representation, in [19] and III.49 of [16], of a Bessel-3 started from $b > 0$ shows that starting R from zero is not a real restriction for the result sought. (Theorem 3 is a discrete analogue of part of Williams' representation.)

Scaling implies that $xR(t/x^2)$ is also Bessel-3 and so $\mathcal{D}(x)/x^2$ has the same distribution for all x. Write $\widetilde{\mathcal{D}}$ for a random variable with distribution $\mathcal{D}(1)$. The next result records rather more than is needed here, and less than is known, about the behaviour of the distribution of $\widetilde{\mathcal{D}}$ at zero and infinity.

Lemma 11 *As $y \to \infty$,*

$$\exp(\pi^2 y/8) P(\widetilde{\mathcal{D}} > y) \to 4/\pi \quad \textit{and} \quad y \exp(y^2/2) P(\widetilde{\mathcal{D}}^{-1} > y^2) \to 2\sqrt{2/\pi}.$$

Proof. This is extracted from Theorem 1 of [8] and Theorem 1.4 of [11], both of which cover other Bessel processes too. $\square$

The first tail estimate on $\widetilde{\mathcal{D}}$ implies that for any $u > 1$ (and $x > 3$)

$$P\left(\frac{\mathcal{D}(x)}{x^2} > u(8/\pi^2)\phi(x)\right) = P\left(\widetilde{\mathcal{D}} > u(8/\pi^2)\phi(x)\right) \le \frac{C}{(\log x)^u}.$$

Considering $\{\mathcal{D}(y^n)/y^{2n} : n \ge 0\}$ with $y > 1$, using Borel-Cantelli and then the monotonicity of $\mathcal{D}$ bounds the $\limsup$ above as required. Turning to the $\liminf$, for any $u > 1$ (and $x > 3$),

$$P\left(\frac{x^2}{\mathcal{D}(x)} > 2u\phi(x)\right) \;=\; P\left(\widetilde{\mathcal{D}}^{-1} > 2u\phi(x)\right) \le C'\sqrt{\frac{1}{2u\phi(x)}}\frac{1}{(\log x)^u},$$

which produces the required lower bound on the $\liminf$.

Pitman's representation of Bessel-3, given in [15], is now used to show both these bounds are tight. This representation writes $R = 2M - B$, where B is a standard Brownian motion and $M(t) = \max\{B(s) : s \le t\}$. Let $\tau(a)$ be the time B first hits a, which is exactly the last time R visits a. Then $\{\tau(x) : x > 0\}$ is a subordinator, with exponent function $\sqrt{2\theta}$, and $\mathcal{D}(x) \leq \tau(x)$. Theorems III.13 and III.14 in [3] describe the growth of sample paths of subordinators. In particular, Theorem III.14 gives, with $\phi(x) = \log\log x$,

$$\liminf_{x\to\infty} \frac{\mathcal{D}(x)}{x^2/\phi(x)} \leq \liminf_{x\to\infty} \frac{\tau(x)}{x^2/\phi(x)} = \frac{1}{2}.$$

This shows the bound on the lim inf is tight.

Finally, let $\mathcal{D}(a;x)$ be the time R spends below $a+x$ after its last visit to a. Pitman's representation implies that $\mathcal{D}(a;x)$ has the same distribution as $\mathcal{D}(x)$. Hence, for any $a \geq 0$, $\mathcal{D}(a;x)$ is distributed like $x^2\widetilde{\mathcal{D}}$ and so, for any $0 < d < 1$ and x large enough,

$$P\left(\frac{\mathcal{D}(a;x)}{x^2} > d\phi(x)\frac{8}{\pi^2}\right) = P\left(\widetilde{\mathcal{D}} > d\phi(x)\frac{8}{\pi^2}\right) \geq \frac{C''}{(\log x)^d}.$$

Pitman's representation also implies that $\mathcal{D}(a;x)$ is independent of $\{R(t) : 0 < t < \tau(a)\}$ and hence that, for any $x > 1$, $\{\mathcal{D}(x^n; x^n(x-1))\}$ are independent as n varies. Furthermore

$$\mathcal{D}(x^{n+1}) \geq \mathcal{D}(x^n; x^n(x-1)).$$

Thus, by Borel Cantelli,

$$\begin{aligned}\limsup_n \frac{\mathcal{D}(x^{n+1})}{(x^{n+1})^2\phi(x^{n+1})} &\geq \left(\frac{x-1}{x}\right)^2 \limsup_n \frac{\mathcal{D}(x^n; x^n(x-1)))}{(x^n(x-1))^2\phi(x^n(x-1))} \\ &\geq \left(\frac{x-1}{x}\right)^2 \frac{8}{\pi^2}.\end{aligned}$$

This holds for arbitrarily large x, which gives the lower bound on the lim sup.

The changes to deal with x going to zero are the obvious ones, with Theorem III.11 in [3] now being deployed to control the subordinator. $\square$

There are some features of the arguments here in common with the discussion of asymptotics associated with the arcsine law in [13]. Also, rather than fixing on constant multiples of ϕ, integral tests for suitable functions could be developed; this is done for the lower bound in [11].

The upper bound on the lim sup and the lower bound on the lim inf are easy consequences of the monotonicity of $\mathcal{D}$, scaling and estimates of the upper tails of $\widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{D}}^{-1}$. These parts of the argument extend immediately to other Bessel processes. In another direction, Pitman's representation, which underpins the rest of the proof, has an analogue for more general Lévy processes conditioned to stay positive, see Section VII.4 in [3] and also [2]; however, these processes do not have the scaling property.

The two constants in Theorem 2 are given by the radii of convergence of the transforms $Ee^{\theta\widetilde{\mathcal{D}}}$ and $Ee^{\theta/\widetilde{\mathcal{D}}}$. In general, for Lévy processes and in the discrete case, it is plausible that, when variances are finite, $\mathcal{D}(x)/x^2$ converges in distribution to $\widetilde{\mathcal{D}}$ as $x \to \infty$. If, in addition, the (upper) tails of $\mathcal{D}(x)/x^2$ and $x^2/\mathcal{D}(x)$ can be controlled, this should lead to U and L in Theorem 1 being given by the corresponding constants in Theorem 2. A Skorokhod-type embedding in R of ζ, or, more precisely, of $V(\zeta)$, is produced in [12], which adds credibility to these speculations.

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