Fixed points of the smoothing transform; the boundary case

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Abstract

Let $A = (A_1, A_2, A_3, \ldots)$ be a random sequence of non-negative numbers that are ultimately zero with $E[\sum A_i] = 1$ and $E[\sum A_i \log A_i] \leq 0$. The uniqueness of the non-negative fixed points of the associated smoothing transform is considered. These fixed points are solutions to the functional equation $\Phi(\psi) = E\left[\prod_i \Phi(\psi A_i)\right]$, where Φ is the Laplace transform of a non-negative random variable. The study complements, and extends, existing results on the case when $E\left[\sum A_i \log A_i\right] < 0$. New results on the asymptotic behaviour of the solutions near zero in the boundary case, where $E\left[\sum A_i \log A_i\right] = 0$, are obtained.

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1 Introduction

Let $A = (A_1, A_2, A_3, \ldots)$ be a random sequence of non-negative numbers that are ultimately zero. Without loss of generality for the results considered, the sequence can, and will, be assumed to be decreasing. Then, there is an almost surely finite N with $A_i > 0$ for $i \leq N$ and $A_i = 0$ otherwise. For any random variable X, let $\{X_i : i\}$ be copies of X, independent of each other and A. A new random variable X^* is obtained as

$$X^* = \sum A_i X_i;$$

unspecified sums and products will always be over i, with i running from 1 to N. Using A in this way to move from X to X^* is called a smoothing transform (presumably because X^* is an 'average' of the copies of X). The random variable W is a fixed point of the smoothing transform when $\sum A_i W_i$ is distributed like W. Here attention is confined to fixed points that are non-negative, that is to $W \geq 0$. This case, though simpler than the one where W is not restricted in this way, still has genuine difficulties; it is intimately connected to limiting behaviours of associated branching processes. For non-negative W, the distributional equation defining a fixed point is expressed naturally in terms of Laplace transforms (with argument $\psi \in (-\infty, \infty)$); it becomes the functional equation (for Φ)

$$\Phi(\psi) = E\left[\prod \Phi(\psi A_i)\right], \tag{1}$$

where Φ is sought in $\mathcal{L}$, the set of Laplace transforms of finite non-negative random variables with some probability of being non-zero. Let $\mathcal{S}(\mathcal{L})$ be the set of solutions to (1) in the set $\mathcal{L}$. The solution corresponds to a variable with finite mean when $-\Phi'(0) < \infty$.

Three fundamental questions concern the existence, uniqueness and asymptotic behaviour near zero of members of $\mathcal{S}(\mathcal{L})$. There is already an extensive literature on these and related questions; see, for example, Kahane and Peyrière (1976), Biggins (1977), Durrett and Liggett (1983), Pakes (1992), Rösler (1992), Biggins and Kyprianou (1997), Liu (1998), Liu (2000), Iksanov and Jurek (2002), Iksanov (2004), Caliebe and Rösler (2003) and Caliebe (2003). The last four references all cite "Biggins and Kyprianou (2001)", which is an earlier version of this paper. Liu (1998) and Liu (2000) contain many further references.

Let the function v be given by

$$v(\theta) = \log E\left[\sum A_i^\theta\right].$$

It is assumed that $v(0) = \log EN > 0$ and that $v(1) = \log E \sum A_i = 0$. Note that, although $v(0) = \log EN$ may be infinite, the definition of A includes the assumption that N itself is finite. Let Z be the point process with points at $\{-\log A_i : A_i > 0\}$ and let μ be its intensity measure. Then

$$e^{v(\theta)} = E \int e^{-\theta x} Z(dx) = \int e^{-\theta x} \mu(dx).$$

Thus $e^{v(\theta)}$ is the Laplace transform of a positive measure and so, in particular, v is convex. Define

$$v'(1) = -\int x e^{-x} \mu(dx),$$

whenever the integral exists (even if v is finite only at $\theta = 1$ so that its derivative has no meaning). Then $v'(1) = E\left[\sum A_i \log A_i\right]$.

Durrett and Liggett (1983) study, fairly exhaustively, the case when N is not random, so that $v(0) = \log N < \infty$, and $v(\gamma) < \infty$ is finite for some $\gamma > 1$; many of their results are extended in Liu (1998) to cases where N is also random, but with moment conditions on the random variables N and $\sum A_i$. Their results deal also with other possibilities, related to those considered here through what they call 'the stable transformation'. The implications of our approach for these other possibilities will be considered in detail elsewhere, and so we do not discuss these extensions here.

The following assumption, the first two parts of which have already been mentioned, will hold throughout.

$$v(0) > 0,\ v(1) = 0 \text{ and } v'(1) \leq 0. \tag{H}$$

Most of the main results also require:

$$v(\theta) < \infty \text{ for some } \theta < 1. \tag{A}$$

In this paper, 'Proposition' is used for results whose proofs can more or less be lifted directly from existing literature. The first of these concerns the existence of solutions and the second concerns uniqueness.

Proposition 1 *When (H) holds $\mathcal{S}(\mathcal{L})$ is non-empty.*

Source: Theorem 3.1 in Liu (1998). □

Proposition 2 *Assume $v'(1) < 0$, (A) holds, $v(\gamma) < \infty$ for some $\gamma > 1$ and $\Phi \in \mathcal{S}(\mathcal{L})$. Then Φ is unique up to a scale factor in its argument.*

Source: Theorem 1.5 of Biggins and Kyprianou (1997). □

The main new feature of this result was that it gave uniqueness when the solution to (1) in $\mathcal{L}$ had an infinite mean. Here the following result, which includes Proposition 2 as a special case, will be proved.

Theorem 3 *Assume that (A) holds and $\Phi \in \mathcal{S}(\mathcal{L})$. Then Φ is unique up to a scale factor in its argument.*

This improves on Proposition 2 in two ways. It dispenses with the requirement that $v(\gamma) < \infty$ for some $\gamma > 1$ and, more significantly, it relaxes $v'(1) < 0$ to $v'(1) \leq 0$. Clearly $v'(1) = 0$ marks the boundary of the cases covered by $v'(1) \leq 0$ and is 'the boundary case' of the title.

Parts of the account in Biggins and Kyprianou (1997) are relevant here. As Proposition 2 hints, that study concerns the 'non-boundary' cases, where $v'(1) < 0$, with additional assumptions. However, many of the proofs there apply more widely, either with no change or with simple modifications. The presentation here aims to make the discussion and the statements of results self-contained, but it will be necessary to consult Biggins and Kyprianou (1997) for details in some proofs.

It is worth indicating how this study connects with some earlier ones related to a branching (standard) Brownian motion with binary splitting at unit rate. Let $N(t)$ be the number of particles in this process at time t with positions given by $(p_1, p_2, \ldots, p_{N(t)})$. Then an instance of the functional equation (1) is

$$\Phi(\psi) = E\left[\prod_{i=1}^{N(t)} \Phi(\psi e^{-\lambda(p_i - c(\lambda)t)})\right],$$

where we can take λ and $c(\lambda)$ such that (H) holds. It turns out (because it is possible to show that the product on the right is a martingale) that when Φ satisfies this for some $t > 0$ it satisfies it for all $t > 0$. Let $\phi(b) = \Psi(e^{-\lambda b})$. Now, with a little effort, by letting $t \downarrow 0$, the functional equation can be converted to the differential equation

$$\frac{1}{2}\phi'' - c(\lambda)\phi' + \phi^2 - \phi = 0,$$

with the boundary conditions $\phi(-\infty) = 0$ and $\phi(\infty) = 1$. For branching Brownian motion, this differential equation is a natural counterpart of the functional equation (1). Classical theory of ordinary differential equations can be used to show that for each $c(\lambda)$ above or at a threshold, there is a unique solution, up to translation. The threshold case corresponds to the boundary case, that is to $v'(1) = 0$. Classical theory also gives information on the behaviour of the solution at infinity, showing, in particular, that the behaviour at infinity of the solution at the threshold value is rather different from the others. The relationships between solutions to this differential equation and martingales associated with the branching Brownian motion are a central strand of the papers by Neveu (1988) and Harris (1999). The properties of the solutions, derived from differential equation theory, form part of the starting point in Neveu's study, as they do in the important earlier ones by McKean (1975) and Bramson (1978, 1983). These studies of branching Brownian motion indicate that the case where $v'(1) = 0$ is likely to be both subtle and important. In contrast to the earlier work, instead of relying on the differential equation theory, Harris (1999) seeks properties of the solutions through probabilistic arguments based on associated martingales. The classical methods for differential equations seem not to extend to (1) but probabilistic arguments do.

Much as in the branching Brownian motion case, the functional equation (1) relates in a natural way to certain martingales in the branching random walk. This relationship and recent results for the martingales, obtained in Biggins and Kyprianou (2004), lead to rather precise information on the behaviour of solutions to (1) near zero when $v'(1) = 0$. To describe this behaviour easily, let L be given by

$$L(\psi) = \frac{1 - \Phi(\psi)}{\psi}, \tag{2}$$

where $\Phi \in \mathcal{S}(\mathcal{L})$. Since $\Phi \in \mathcal{L}$, L is a Laplace transform of a measure on $(0, \infty)$ and hence is decreasing in ψ, and then $L(0+)$ is finite exactly when the random variable corresponding to Φ has a finite mean. Hence, the next theorem implies that in the boundary case every solution to (1) has an infinite mean. A new assumption occurs here:

$$v''(1) = \textstyle\int x^2 e^{-x} \mu(dx) < \infty. \tag{V}$$

Theorem 4 *Assume $v'(1) = 0$, (A) and (V) hold and $\Phi \in \mathcal{S}(\mathcal{L})$. Then $(-\log \psi)^{-1} L(\psi)$ has a limit as $\psi \downarrow 0$ and the limit is strictly positive but may be infinite.*

The assumption (A) implies that $\int_{-\infty}^{0} x^2 e^{-x} \mu(dx) < \infty$, which is 'half' of (V), and so, in the statement of the previous theorem and the next one, (V) could have been rephrased to reflect this. However, (V) is used in intermediate results where (A) is not imposed.

To say more about the limit in Theorem 4, we need to introduce the following non-negative random variables:

$$G_1 = -\sum A_i \log A_i I(A_i < 1) \text{ and } \Gamma^{(s)} = \sum A_i I(A_i > e^{-s}).$$

Note that $\Gamma^{(s)} \uparrow \Gamma^{(\infty)} = \sum A_i$ as $s \uparrow \infty$. Also, let $\phi(x) = \log \log \log x$, $L_1(x) = (\log x)\phi(x)$, $L_2(x) = (\log x)^2 \phi(x)$, $L_3(x) = (\log x)/\phi(x)$ and $L_4(x) = (\log x)^2/\phi(x)$; the key point about these functions is that L_1 and L_3 are similar to each other and to $\log x$ and L_2 and L_4 are similar to each other and to $(\log x)^2$. Hence the moment conditions in parts (a) and (b) of the next theorem are close to each other, but there are cases in between. The moment conditions in (a) are quite mild and so, informally, (a) gives the typical behaviour.

Theorem 5 *Assume $v'(1) = 0$, (A) and (V) hold and $\Phi \in \mathcal{S}(\mathcal{L})$.*
(a) If both

$$E[G_1 L_1(G_1)] < \infty \textit{ and } E[\Gamma^{(\infty)} L_2(\Gamma^{(\infty)})] < \infty$$

then

$$\lim_{\psi \downarrow 0} (-\log \psi)^{-1} L(\psi) \in (0, \infty).$$

(b) If

$$E\left[G_1 L_3(G_1)\right] = \infty \quad \textit{or} \quad E\left[\Gamma^{(s)} L_4(\Gamma^{(s)})\right] = \infty \textit{ for some } s$$

then

$$\lim_{\psi \downarrow 0} (-\log \psi)^{-1} L(\psi) = \infty.$$

Under the assumption that N and $\Gamma^{(\infty)}$ both have a finite $1+\delta$ moment for some $\delta > 0$, Theorem 1.4 of Liu (1998) applies to show that $(-\log \psi)^{-1} L(\psi)$ converges to a limit in $(0, \infty)$ as $\psi \downarrow 0$. (In Liu's notation, $\alpha = 1$, $\rho'(\alpha) = 0$ and p is a positive constant.) Thus, for the case studied here, Theorems 4 and 5 improve on Theorem 1.4 of Liu (1998).

We finish this introduction with an overview of the rest of the paper. The relationships between the functional equation and martingales arising in an associated branching random walk are described in the next section. These allow the functional equation (1) to be transformed to another of the same form but satisfying stronger assumptions. This reduction is described in Section 3 and used to prove Theorem 3 from Proposition 2. Section 4 illustrates further the usefulness of this reduction and prepares the ground for the proofs of Theorems 4 and 5, which are contained in the final two sections.

2 Multiplicative martingales

There is a natural (one to one) correspondence, already hinted at, between the framework introduced and the branching random walk, a connection that is the key to some of the proofs. Let the point process Z (with points at $\{-\log A_i : A_i > 0\}$) be used to define a branching random walk in the following way. The process starts with a single particle located at the origin. This particle produces daughter particles, with positions given by a real number, to give the first generation. In a similar way, these first generation particles produce daughter particles, to give the second generation, and so on. More precisely, given the development of the process to the nth generation, each nth generation particle reproduces independently with the size and the positions, relative to the parent's, of each family in the $(n+1)$th generation being given by a copy of Z. Ignoring positions gives a Galton-Watson process with (almost surely finite) family size N. People are labelled by their ancestry (the Ulam-Harris labelling) and the generation of u is $|u|$. Let z_u be the position of u, so that $\{z_u : |u| = 1\}$ is a copy of $\{-\log A_i : A_i > 0\}$. Then the assumption (H) translates to:

$$\int \mu(dx) > 1,\ \int e^{-x}\mu(dx) = 1 \text{ and } \int xe^{-x}\mu(dx) \geq 0. \tag{H}$$

Proofs here, and in Biggins and Kyprianou (1997), rely on the nth generation of the branching random walk going to infinity with n. To give the result needed, let B_n be the position of the left-most person in the nth generation, that is $B_n = \inf\{z_u : |u| = n\}$, which is taken to be infinite when the branching process has already died out by then.

Proposition 6 *The assumption (H) is enough to ensure that $B_n \to \infty$ almost surely.*

Source: Theorem 3 of Biggins (1998) or Lemma 7.2 in Liu (1998). □

The next result is easy to establish. It is natural to call the martingales it describes multiplicative martingales.

Proposition 7 *Let $\Phi \in \mathcal{S}(\mathcal{L})$, so that Φ solves (1) and $0 < \Phi(\psi) < 1$ for $\psi > 0$. Then, for each $\psi > 0$*

$$\prod_{|u|=n} \Phi\left(\psi e^{-z_u}\right) \quad \text{for } n = 0, 1, 2, 3, \ldots,$$

is a bounded martingale, which converges in mean and almost surely to a limit $M(\psi)$. Furthermore, $EM(\psi) = \Phi(\psi)$, and so $P(M(\psi) < 1) > 0$ for all $\psi > 0$.

Source: Theorem 3.1 and Corollary 3.2 in Biggins and Kyprianou (1997).

In random walk theory the ladder height, the first point in $(0, \infty)$ reached by the random walk, is an important concept. From a random walk's trajectory, a sequence of successive independent identically distributed ladder heights can be constructed, each new one arising as the first overshoot of the previous maximum. Analogous ideas are important here.

For the branching random walk corresponding to A let

$$\mathcal{C} = \{u : z_u > 0 \text{ but } z_v \leq 0 \text{ for } v < u\}, \tag{3}$$

where $v < u$ means v is an ancestor of u. Hence $\mathcal{C}$ identifies the individuals who are the first in their lines of descent to be to the right of 0. The collection $\{z_u : u \in \mathcal{C}\}$ has the same role here as the first ladder height in a random walk. This motivates the next construction.

Starting from the initial ancestor, follow a line of descent down to its first member to the right of 0; doing this for all lines of descent produces $\mathcal{C}$. Regard the members of $\mathcal{C}$ as the children of the initial ancestor, rather than simply descendants; the resulting point process of children's positions, $\{z_u : u \in \mathcal{C}\}$, is concentrated on $(0, \infty)$ by arrangement. Now, pick a member of $\mathcal{C}$; follow a line of descent from this individual down to the first member to the right of the member picked; doing this for all lines of descent from the member picked produces a copy of $\{z_u : u \in \mathcal{C}\}$. This can be done for each member of $\mathcal{C}$ to produce a family for each of them, giving a 'second generation'. In the same way, families can be identified for these 'second generation' individuals and so on. The positions of individuals in this embedded process, which are all in $(0, \infty)$, can now be interpreted as birth times. The result is a general branching process, also called a Crump-Mode-Jagers (CMJ) process, constructed from individuals and their positions in the branching random walk.

It is possible to describe explicitly which individuals occur in the embedded process. In the branching random walk, for $t \geq 0$ let

$$\mathcal{C}(t) = \{u : z_u > t \text{ but } z_v \leq t \text{ for } v < u\},$$

so that $\mathcal{C}(0) = \mathcal{C}$, and let $\mathcal{C}(t)$ be the initial ancestor for $t < 0$. The individuals in the branching random walk that occur in the CMJ process are exactly those in $\mathcal{C}(t)$ as t varies. The variable t can be interpreted as time. Then $\mathcal{C}(t)$ is what is called the coming generation for the CMJ process; it consists of the individuals born after t whose mothers are born no later than t. This whole construction is discussed more formally in Section 8 of Biggins and Kyprianou (1997); the relevant aspect is summarised in the next result. Naturally, the process constructed is called the embedded CMJ process.

Proposition 8 *The individuals $\{u : u \in \mathcal{C}(t) \text{ for some } t\}$, with the mother of u defined to be u's closest ancestor in the collection and u's birth time being z_u, form a general (CMJ) branching process with reproduction point process $\{z_u : u \in \mathcal{C}\}$.*

The first part of the next Proposition introduces multiplicative martingales like those defined in Proposition 7, but with the products taken over $\mathcal{C}(t)$. These martingales play a further part in the development here. They lead immediately to a functional equation for Φ of the form (1) but with a different A, which is the Proposition's final assertion. This transformation of the problem is examined further in the next section.

Proposition 9 *Let $\Phi \in \mathcal{S}(\mathcal{L})$. For $-\infty < t < \infty$, let*

$$M_t(\psi) = \prod_{u \in \mathcal{C}(t)} \Phi(\psi e^{-z_u}).$$

For each $\psi \geq 0$, $M_t(\psi)$ is a bounded martingale. In particular, Φ satisfies

$$\Phi(\psi) = E\left[\prod_{u \in \mathcal{C}} \Phi(\psi e^{-z_u})\right].$$

Source: Theorem 6.2 and Lemma 8.1 in Biggins and Kyprianou (1997). □

3 Reduction of the functional equation.

A major element in the approach here is the reduction of certain cases to simpler ones with stronger assumptions; this reduction is made precise in the next result. Given A, let A^* be the numbers $\{e^{-z_u} : u \in \mathcal{C}\}$, defined through (3), in decreasing order. Objects derived from A, like N and μ, have counterparts for A^*, denoted by N^*, μ^* and so on. The reproduction point process of the embedded CMJ process, introduced in the previous section, is $\{z_u : u \in \mathcal{C}\}$, which has intensity measure μ^*. Hence the next theorem can easily be reinterpreted to give properties of μ^*.

Theorem 10 *Let* $\Phi \in \mathcal{S}(\mathcal{L})$*.*
(a) Then Φ *is also a solution to*

$$\Phi(\psi) = E\left[\prod \Phi(\psi A_i^*)\right]$$

and $\max A_i^* < 1$*.*
(b) If (H) holds for A *then (H) holds for* A^**. Furthermore,* $v^{*\prime}(1) < 0$*.*
(c) If (A) holds for A *then it also holds for* A^* *with the same* θ*.*
(d) If $P(N < \infty) = 1$ *then* $P(N^* < \infty) = 1$*.*

It is worth stressing that not all properties transfer exactly; for example, (V) for A does not imply (V) for A^*.

Before giving the proof, we need the following result, linking quantities of interest to expectations for random walk.

Proposition 11 *Let* $S_0 = 0$ *and let* S_n *be the sum of* n *independent identically distributed variables with law* $e^{-x}\mu(dx)$*. Then*

$$E\sum_{|u|=n} e^{-z_u} f(z_v : v \leq u) = E(f(S_k : k \leq n))$$

for all (measurable) functions f*. In particular, taking* $n = 1$ *and writing in terms of* A*,*

$$E\sum A_i f(-\log A_i) = E(f(S_1)).$$

Source: Lemma 4.1(iii) of Biggins and Kyprianou (1997); see also p289 of Durrett and Liggett (1983) as well as Lemma 1 of Bingham and Doney (1975). □

Notice that, by (H), $ES_1 = -v'(1) \geq 0$. Hence the random walk $S = \{S_n : n \geq 0\}$ has a non-negative drift. Let $\tau = \inf\{n \geq 0 : S_n \in (0, \infty)\}$, which must be finite almost surely because $ES_1 \geq 0$. Then S_τ is the first strict increasing ladder height of S.

Proof of Theorem 10. Part (a) is just a restatement of the final part of Proposition 9 and the fact that by definition all terms in $\{z_u : u \in \mathcal{C}\}$ are strictly positive.

Since

$$\exp(v^*(\theta)) = E\left[\sum (A_i^*)^\theta\right] = E\sum_{u\in\mathcal{C}} e^{-\theta z_u},$$

to prove (b), that is that (H) holds for v^* with $v^{*\prime}(1) < 0$, we must show that

$$E|\mathcal{C}| > 1, \quad E\sum_{u\in\mathcal{C}} e^{-z_u} = 1 \text{ and } E\sum_{u\in\mathcal{C}} z_u e^{-z_u} > 0.$$

The last of these is immediate from the first and the positivity of $\{z_u : u \in \mathcal{C}\}$. For the first two note, using Proposition 11, that

$$\begin{aligned}
E\sum_{u\in\mathcal{C}} f(z_u)e^{-z_u} &= \sum_{n\geq 1} E\sum_{|u|=n} I\left(z_u > 0, z_v \leq 0 \text{ for all } v < u\right) f(z_u)e^{-z_u} \\
&= \sum_{n\geq 1} E\left[I\left(S_n > 0, S_k \leq 0 \text{ for all } k < n\right) f(S_n)\right] \\
&= E\left[I(\tau < \infty) f(S_\tau)\right].
\end{aligned}$$

In particular,

$$E\sum_{u\in\mathcal{C}} e^{-z_u} = P\left(\tau < \infty\right) = 1$$

and then

$$E|\mathcal{C}| > E\sum_{u\in\mathcal{C}} e^{-z_u} = 1.$$

For (c) note that

$$E\left[\sum_{u\in\mathcal{C}} e^{-\theta z_u}\right] = E\left[\sum_{u\in\mathcal{C}} e^{(1-\theta)z_u}e^{-z_u}\right] = Ee^{(1-\theta)S_\tau}.$$

Thus the required finiteness reduces to the ladder height S_τ having a suitable exponential tail. Now

$$\infty > e^{v(\theta)} = \int e^{-\theta x}\mu(dx) = \int e^{(1-\theta)x}e^{-x}\mu(dx) = E[e^{(1-\theta)S_1}]$$

and so the tails of the increment distribution of the random walk decay exponentially. This implies, by standard random walk theory, in particular, XII(3.6a) in Feller (1971), that the same is true of S_τ.

Recall that B_n is the position of the left-most person in the nth generation. By Proposition 6, $B_n \to \infty$ and so $\mathcal{C}$ is contained entirely within some finite number of generations. Since N, the family size, is finite this forces $|\mathcal{C}|$ to be finite, giving (d). $\square$

Proof of Theorem 3. When $\max A_i < 1$ it is clear that $v(\gamma) < \infty$ for some $\gamma > 1$ and that $v'(1) < 0$. Hence, Theorem 10 reduces the cases being considered in Theorem 3 to those covered by Proposition 2. $\square$

The proof of Proposition 2 relies heavily on results in Nerman (1981) where, μ^* is assumed to be non-lattice, but Nerman rightly says 'all results could be modified to the lattice case'. The details of the lattice case can be seen in Gatzouras (2000).

4 Slow variation and its consequences

To prove Theorems 4 and 5 a little more information about the behaviour of the multiplicative martingales is needed.

Proposition 12 *Let $\Phi \in \mathcal{S}(\mathcal{L})$ and L be given by (2). Then $L(\psi)$ is slowly varying as $\psi \downarrow 0$.*

Source: Theorem 1.4 of Biggins and Kyprianou (1997) when $v'(1) < 0$; Theorem 2 of Kyprianou (1998) when $v'(1) = 0$.

However, it is worth noting that Theorem 10 transforms cases where $v'(1) = 0$ into ones where $v'(1) < 0$ and then Theorem 1.4 of Biggins and Kyprianou (1997) applies. In this way the use of Theorem 2 of Kyprianou (1998) could be circumvented. $\square$

The next result is a routine extension of what is already known.

Proposition 13 *Let $\Phi \in \mathcal{S}(\mathcal{L})$, L be given by (2) and $M(\psi)$ be the limit introduced in Proposition 7.*
(a)

$$\lim_{n\to\infty} \sum_{|u|=n} e^{-z_u} L(e^{-z_u}) = -\log M(1).$$

(b) $M(\psi) = M(1)^\psi$ and so $\Phi(\psi) = E[e^{\log M(1)\psi}]$.

Proof. These results are proved in Lemmas 5.1 and 5.2 of Biggins and Kyprianou (1997). Those proofs work here, but now use Propositions 6 and 12 for the facts that $B_n \to \infty$ and L is slowly varying. $\square$

Proposition 14 *Assume $v'(1) < 0$, (A) holds, $v(\gamma) < \infty$ for some $\gamma > 1$ and $\Phi \in \mathcal{S}(\mathcal{L})$. Then*

$$\lim_{t\to\infty} L(e^{-t}) \sum_{u\in\mathcal{C}(t)} e^{-z_u} = W, \tag{4}$$

where W has Laplace transform Φ.

Source: Theorem 8.6 in Biggins and Kyprianou (1997). $\square$

Theorem 15 *Assume (A) holds and $\Phi \in \mathcal{S}(\mathcal{L})$. Then the conclusion of Proposition 14 holds.*

Proof. Proposition 8 describes a CMJ embedded in the original branching random walk. Theorem 10 shows that the embedded CMJ, viewed as a branching random walk with only positive steps, satisfies all the conditions of Proposition 14. The conclusion is then that (4) holds for the embedded CMJ process of this branching random walk with only positive steps, but in such a case the embedded process is identical to the original one. $\square$

Further argument, of the kind in Biggins and Kyprianou (1997), shows that $W = -\log M(1)$, but this connection is not needed for the subsequent arguments.

5 The derivative martingale

In this section will consider only the case where $v'(0) = \int xe^{-x}\mu(dx) = 0$, that is the boundary case. We will describe some properties of a martingale that is intimately related to the properties of the functional equation in this case.

Let

$$\partial W_n = \sum_{|u|=n} z_u e^{-z_u};$$

then it is straightforward to check that ∂W_n is a martingale. It is called the derivative martingale because its form can be derived by differentiating $\sum_{|u|=n} e^{-\theta z_u - nv(\theta)}$, which is also a martingale, with respect to θ and then setting θ to one. The martingale ∂W_n has been considered in Kyprianou (1998) and Liu (2000) and its analogue for branching Brownian motion has been discussed by several authors — Neveu (1988), Harris (1999) and Kyprianou (2004) for example.

The derivative martingale is one of the main examples in Biggins and Kyprianou (2004), where general results on martingale convergence in branching processes are discussed.

Proposition 16 *When $v'(1) = 0$ and (V) holds, the martingale ∂W_n converges to a finite non-negative limit, Δ, almost surely. Then*

$$\Delta = \sum_{|u|=1} e^{-z_u}\Delta_u,$$

where, given the first generation, for each u such that $|u| = 1$, Δ_u are independent copies of Δ. Furthermore, $P(\Delta = 0)$ is either equal to the extinction probability or equal to one.

Source: Theorem 4.1 in Biggins and Kyprianou (2004). □

This result shows that the transform of Δ satisfies (1) and will have a transform in $\mathcal{L}$ when Δ is not identically zero. Whether the martingale limit Δ is zero or not is related to the behaviour of the solution to (1) near the origin. The precise relationship is formulated in the next theorem, the proof of which is deferred to Section 6.

Theorem 17 *Suppose $v'(1) = 0$, (A) and (V) hold and $\Phi \in \mathcal{S}(\mathcal{L})$. Then $P(\Delta > 0) > 0$ if and only if*

$$\lim_{\psi \downarrow 0} \left((-\log \psi)^{-1} L(\psi) \right) = c \in (0, \infty); \tag{5}$$

furthermore $P(\Delta = 0) = 1$ if and only if $(-\log \psi)^{-1} L(\psi) \to \infty$ as $\psi \downarrow 0$. In fact, (A) is not needed for the 'if' parts here.

Proof of Theorem 4. This result is contained in Theorem 17. □

Information on when Δ is not zero, and when it is, is given in the next result, with the notation used in Theorem 5.

Proposition 18 *Assume $v'(1) = 0$ and that (V) holds.*
(a) If both $E[G_1 L_1(G_1)] < \infty$ and $E[\Gamma^{(\infty)} L_2(\Gamma^{(\infty)})] < \infty$ then Δ is not identically zero.
(b) If $E\left[G_1 L_3(G_1)\right] = \infty$ or $E\left[\Gamma^{(s)} L_4(\Gamma^{(s)})\right] = \infty$ for some $s > 0$ then $\Delta = 0$ almost surely.

Source: Theorem 4.1 of Biggins and Kyprianou (2004). □

Proof of Theorem 5. Combine Theorem 17 and Proposition 18. □

Some results on the relationship between the limiting behaviour in (5), the limit Δ and the uniqueness of the solution to (1) have been obtained previously, in Kyprianou (1998) and Liu (2000). Those studies approach the convergence of ∂W_n and uniqueness through (5). However, that approach to uniqueness is limited because Theorem 17 and Proposition 18 show that the asymptotic (5) does not always hold.

6 Proof of Theorem 17

Most of the work is done by a preliminary lemma. Its proof borrows heavily from the proof of Theorem 8.6 in Biggins and Kyprianou (1997).

Lemma 19 *Suppose $v'(1) = 0$, and that (A) and (V) hold. Then*

$$t \sum_{u \in \mathcal{C}(t)} e^{-z_u} \to \Delta$$

almost surely, as $t \to \infty$.

Proof. Note first that, for $x \geq 1$ and any $\epsilon \in (0,1)$, $\epsilon x \leq e^{\epsilon(x-1)}$ and thus, for $u \in \mathcal{C}(t)$ and $t \geq 1$,

$$z_u/t \leq \epsilon^{-1} e^{\epsilon(z_u - t)}. \tag{6}$$

Take $\epsilon = 1 - \theta$, where θ comes from (A). Then, a routine application of Theorem 6.3 in Nerman (1981), following closely the corresponding calculation in the proof of Theorem 8.6 in Biggins and Kyprianou (1997), shows that

$$\lim_{c \to \infty} \lim_{t \to \infty} \frac{\sum_{u \in \mathcal{C}(t)} e^{-(1-\epsilon)(z_u - t)} I(z_u > t + c)}{\sum_{u \in \mathcal{C}(t)} e^{-(z_u - t)}} = 0 \quad \text{a.s.}, \tag{7}$$

when $\mathcal{C}(t)$ is not eventually empty — some further details of the calculation needed to establish (7) are given at the end of this proof.

Now, for $t \geq 1$ and $\mathcal{C}(t)$ non-empty,

$$\begin{aligned} 1 \leq \frac{\sum_{u \in \mathcal{C}(t)} z_u e^{-z_u}}{\sum_{u \in \mathcal{C}(t)} t e^{-z_u}} &= \frac{\sum_{u \in \mathcal{C}(t)} z_u e^{-(z_u - t)}}{\sum_{u \in \mathcal{C}(t)} t e^{-(z_u - t)}} \\ &\leq \frac{t + c}{t} + \frac{\sum_{u \in \mathcal{C}(t)} (z_u/t) e^{-(z_u - t)} I(z_u > t + c)}{\sum_{u \in \mathcal{C}(t)} e^{-(z_u - t)}}. \end{aligned}$$

Applying (6) to the final term gives

$$1 \leq \frac{\sum_{u \in \mathcal{C}(t)} z_u e^{-z_u}}{\sum_{u \in \mathcal{C}(t)} t e^{-z_u}} \leq \frac{t + c}{t} + \epsilon^{-1} \frac{\sum_{u \in \mathcal{C}(t)} e^{-(1-\epsilon)(z_u - t)} I(z_u > t + c)}{\sum_{u \in \mathcal{C}(t)} e^{-(z_u - t)}}$$

and right hand side tends to one as t and then c tend to infinity by (7). The proof is completed by noting that

$$\sum_{u \in \mathcal{C}(t)} z_u e^{-z_u} \to \Delta$$

almost surely as $t \to \infty$, by Theorem 4.2 in Biggins and Kyprianou (2004).

It remains to consider (7) further. Note first that, since the ratio is monotonically decreasing in c, the outer limit can be confined to the rationals. A little more discussion of the general (CMJ) branching process is needed to expose the main issues remaining. As explained in Proposition 8, the CMJ's reproduction point process is $\{z_u : u \in \mathcal{C}\}$ with intensity measure μ^*, and the individuals in the original process who occur in the embedded process are those in $\mathcal{T}$, where $\mathcal{T} = \cup_{t \geq 0} \mathcal{C}(t)$. For these individuals, z_u can be interpreted as a birth time. We also need the notion of a characteristic, which is a mechanism for counting the population. Each individual in the CMJ has associated with it an independent copy of some function χ, and this function measures the contribution of the individual, as she grows older, to a count of the process. These functions are zero for negative ages. The CMJ process counted by χ, which is denoted by ζ^χ, is defined at t by

$$\zeta_t^\chi = \sum_{u \in \mathcal{T}} \chi_u(t - z_u).$$

The top line in (7) is the CMJ processes counted using the characteristic

$$\psi(a) = I\{a > 0\} \sum_{u \in \mathcal{C}} e^{-(1-\epsilon)(z_u - a)} I(z_u > a + c),$$

whilst the denominator is the CMJ process counted using the characteristic

$$\chi(a) = I\{a > 0\} \sum_{u \in \mathcal{C}} e^{-(z_u - a)} I(z_u > a).$$

To complete the calculation for the case when μ^* is non-lattice, we need the following result, which is contained in Theorem 6.3 of Nerman (1981) and can also be found in Theorem X.5.1 in Asmussen and Hering (1983).

Proposition 20 *Suppose that:*
(i) μ^* *is non-lattice and* $\int e^{-a}\mu^*(da) = 1$*;*
(ii) there is a $\beta < 1$ *such that* $\int e^{-\beta a}\mu^*(da) < \infty$*;*
(iii) ψ *and* χ *are two characteristics with* $E \sup e^{-\beta t}\psi(t)$ *and* $E \sup e^{-\beta t}\chi(t)$ *both finite.*
Then, on the survival set of the process,

$$\frac{\zeta_t^\psi}{\zeta_t^\chi} \to \frac{\int_0^\infty e^{-t} E\psi(t) dt}{\int_0^\infty e^{-t} E\chi(t) dt} \quad a.s. \quad as \quad t \to \infty.$$

We now apply Proposition 20 with $\beta = 1 - \epsilon = \theta$, where θ comes from (A). By (b) and (c) of Theorem 10,

$$\int e^{-a}\mu^*(da) = 1 \quad \text{and} \quad \int e^{-\theta a}\mu^*(da) < \infty.$$

It remains to check the supremum condition on the characteristics. For ψ, note that for $a > 0$

$$\begin{aligned} e^{-a\theta}\psi(a) &= e^{-a\theta}\sum_{u\in\mathcal{C}} e^{-\theta(z_u-a)} I(z_u > a+c) \\ &= \sum_{u\in\mathcal{C}} e^{-\theta z_u} I(z_u > a+c) \le \sum_{u\in\mathcal{C}} e^{-\theta z_u}, \end{aligned}$$

which is independent of a and has expectation $\int e^{-\theta a}\mu^*(da)$, which is finite. Similarly, again for $a > 0$,

$$\begin{aligned} e^{-a\theta}\chi(a) &= e^{-a\theta}\sum_{u\in\mathcal{C}} e^{-(z_u-a)} I(z_u > a) \\ &\le e^{-a\theta}\sum_{u\in\mathcal{C}} e^{-\theta(z_u-a)} I(z_u > a) \le \sum_{u\in\mathcal{C}} e^{-\theta z_u}. \end{aligned}$$

Thus

$$\lim_{t\to\infty} \frac{\sum_{u\in\mathcal{C}(t)} e^{-(1-\epsilon)(z_u-t)} I(z_u > t+c)}{\sum_{u\in\mathcal{C}(t)} e^{-(z_u-t)}} = \frac{\int_0^\infty e^{-\epsilon a}\left(\int_{a+c}^\infty e^{-\sigma(1-\epsilon)}\mu^*(d\sigma)\right)da}{\int_0^\infty\int_a^\infty e^{-\sigma}\mu^*(d\sigma)da}$$

almost surely. The denominator is $\int_0^\infty \sigma e^{-\sigma}\mu^*(d\sigma)$, which is finite and the numerator goes to zero as c goes to infinity.

The lattice case is handled similarly, but draws on Theorem 5.2 in Gatzouras (2000). $\square$

Proof of Theorem 17. Suppose that $((-\log\psi)^{-1}L(\psi))$ has a limit ℓ as $\psi \downarrow 0$. Then, when ℓ is finite,

$$\begin{aligned} \left|\sum_{|u|=n} e^{-z_u}L(e^{-z_u}) - \sum_{|u|=n} \ell z_u e^{-z_u}\right| &\le \left(\sup_{|u|=n}\left|\frac{L(e^{-z_u})}{z_u} - \ell\right|\right)\left(\sum_{|u|=n} z_u e^{-z_u}\right) \\ &\le \left(\sup_{z\ge B_n}\left|\frac{L(e^{-z})}{z} - \ell\right|\right)\partial W_n \end{aligned}$$

which goes to zero as $n\to\infty$, using Propositions 6 and 16. Thus, using Propositions 13(a) and 16,

$$-\log M(1) = \lim_{n\to\infty}\sum_{|u|=n} e^{-z_u}L(e^{-z_u}) = \ell \lim_{n\to\infty}\sum_{|u|=n} z_u e^{-z_u} = \ell\Delta.$$

Similarly, if $\ell = \infty$, for any finite C

$$-\log M(1) = \lim_{n\to\infty}\sum_{|u|=n} e^{-z_u}L(e^{-z_u}) \ge \lim_{n\to\infty}\sum_{|u|=n} C z_u e^{-z_u} = C\Delta.$$

By Proposition 13(b), $-\log M(1)$ is finite and not identically zero. Hence, when $\ell < \infty$, Δ must also be finite and not identically zero. In contrast, when $\ell = \infty$, Δ must be identically zero, This part of the argument is based on the proof of Theorem 2.5 of Liu (2000); see also Theorem 3 of Kyprianou (1998).

Turning to the 'only if' claims, let Δ be the limit of ∂W_n and let $\Phi \in \mathcal{S}(\mathcal{L})$. Then, by Theorem 15 and Lemma 19

$$\frac{\Delta}{W} = \lim_{t\uparrow\infty} \frac{t\sum_{u\in\mathcal{C}(t)} e^{-z_u}}{L(e^{-t})\sum_{u\in\mathcal{C}(t)} e^{-z_u}} = \lim_{t\uparrow\infty} \frac{t}{L(e^{-t})} = \lim_{t\uparrow\infty} \frac{te^{-t}}{1-\Phi(e^{-t})},$$

which must be a (non-random) constant. The constant is only zero when Δ is identically zero; otherwise, (5) holds. $\square$

The first half of the proof just given is unnecessary when (A) holds, since the second half actually gives the claimed equivalence. Hence this treatment could have omitted Propositions 12 and 13 by sacrificing the last assertion in Theorem 17.

The idea that the convergence described in Lemma 19 produces information on the asymptotics of the functional equation occurs, in the branching Brownian motion context with non-trivial Δ, in Kyprianou (2004). It is also worth noting that Lemma 19 provides a Seneta-Heyde norming for the Nerman martingale $\sum_{u\in\mathcal{C}(t)} e^{-z_u}$ associated with the particular CMJ process arising here. The existence of such a norming in general is covered by Theorem 7.2 of Biggins and Kyprianou (1997). The special structure here means that the slowly varying function in the general theorem is the logarithm.

Lyons (1997) shows that when $v'(1) = 0$ the non-negative martingale $W_n = \sum_{|u|=n} e^{-z_u}$ converges almost surely to zero. In the same spirit as Theorem 1.2 in Biggins and Kyprianou (1997), it is natural to wonder whether there are constants c_n such that W_n/c_n converges. In Biggins and Kyprianou (1997), the approach to this question, which we have not been able to settle in the present context, needs a 'law of large numbers' which would say, roughly, $W_{n+1}/W_n \to 1$ in probability.

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