Fixed points of the smoothing transform; the boundary case*

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Abstract

Let $A = (A_1, A_2, A_3, \ldots)$ be a random sequence of non-negative numbers that are ultimately zero with $E[\sum A_i] = 1$ and $E[\sum A_i \log A_i] \leq 0$. The uniqueness of the non-negative fixed points of the associated smoothing transform is considered. These fixed points are solutions to the functional equation $\Phi(\psi) = E[\prod_i \Phi(\psi A_i)]$, where Φ is the Laplace transform of a non-negative random variable. The study complements, and extends, existing results on the case when $E[\sum A_i \log A_i] < 0$. New results on the asymptotic behaviour of the solutions near zero in the boundary case, where $E[\sum A_i \log A_i] = 0$, are obtained.

Running head: SMOOTHING TRANSFORM

1 Introduction

Let $A = (A_1, A_2, A_3, \ldots)$ be a random sequence of non-negative numbers that are ultimately zero. Without loss of generality for the results considered, the sequence can, and will, be assumed to be decreasing. Then, there is an almost surely finite N with $A_i > 0$ for $i \leq N$ and $A_i = 0$ otherwise. For any random variable X, let $\{X_i : i\}$ be copies of X, independent of each other and A. A new random variable X^* is obtained as

$$X^* = \sum A_i X_i;$$

unspecified sums and products will always be over i, with i running from 1 to N. Using A in this way to move from X to X^* is called a smoothing transform (presumably because X^* is an 'average' of the copies of X). The random variable W is a fixed point of the smoothing transform when $\sum A_i W_i$ is distributed like W. Here attention is confined to fixed points that are non-negative, that is to $W \geq 0$. This case, though simpler than the one where W is not restricted in this way, still has genuine difficulties; it is intimately connected to limiting behaviours of associated branching processes. For non-negative W, the distributional equation defining a fixed point is expressed naturally in terms of Laplace transforms; it becomes the functional equation (for Φ)

$$\Phi(\psi) = E\left[\prod \Phi(\psi A_i)\right], \tag{1}$$

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where Φ is sought in $\mathcal{L}$, the set of Laplace transforms of finite non-negative random variables with some probability of being non-zero. Let $\mathcal{S}(\mathcal{L})$ be the set of solutions to (1) in the set $\mathcal{L}$. The solution corresponds to a variable with finite mean when $-\Phi'(0) < \infty$.

Three fundamental questions concern the existence, uniqueness and asymptotic behaviour near zero of members of $\mathcal{S}(\mathcal{L})$. There is already an extensive literature on these and related questions; see, for example, Kahane and Peyrière (1976), Biggins (1977), Durrett and Liggett (1983), Pakes (1992), Rösler (1992), Biggins and Kyprianou (1997), Liu (1998), Liu (2000), Iksanov and Jurek (2002), Iksanov (2002), Caliebe and Rösler (2003) and Caliebe (2003). The last four references all cite Biggins and Kyprianou (2001b), which is an earlier version of this paper. Liu (1998) and Liu (2000) contain many further references.

Let the function v be given by

$$v(\theta) = \log E\left[\sum A_i^\theta\right].$$

It is assumed that $v(0) = \log EN > 0$ and that $v(1) = E\sum A_i = 1$. Note that, although $v(0) = \log EN$ may be infinite, the definition of A includes the assumption that N itself is finite. Let Z be the point process with points at $\{-\log A_i : A_i > 0\}$ and let μ be its intensity measure. Then

$$e^{v(\theta)} = E\int e^{-\theta x} Z(dx) = \int e^{-\theta x}\mu(dx).$$

Thus $e^{v(\theta)}$ is the Laplace transform of a positive measure and so, in particular, v is convex. Define

$$v'(1) = -\int x e^{-x}\mu(dx),$$

whenever the integral exists (even if v is finite only at $\theta = 1$ so that its derivative has no meaning). Then $v'(1) = -E\left[\sum A_i \log A_i\right]$.

Durrett and Liggett (1983) study, fairly exhaustively, the case when N is not random, so that $v(0) = \log N < \infty$, and $v(\gamma) < \infty$ is finite for some $\gamma > 1$; many of their results are extended in Liu (1998) to cases where N is also random, but with moment conditions on the random variables N and $\sum A_i$. Their results deal also with other possibilities, related to those considered here through what they call 'the stable transformation'. The implications of our approach for these other possibilities will be considered in detail elsewhere, and so we do not discuss these extensions here.

The following assumption, the first two parts of which have already been mentioned, will hold throughout.

(H) $$v(0) > 0,\ v(1) = 0 \text{ and } v'(1) \leq 0.$$

Most of the main results also require:

(A) $$v(\theta) < \infty \text{ for some } \theta < 1.$$

In this text 'Proposition' is used for results whose proofs can more or less be lifted directly from existing literature. The first of these concerns the existence of solutions and the second concerns uniqueness.

Proposition 1 *When (H) holds $\mathcal{S}(\mathcal{L})$ is non-empty.*

Source: Theorem 3.1 in Liu (1998). □

Proposition 2 *Assume $v'(1) < 0$, (A) holds, $v(\gamma) < \infty$ for some $\gamma > 1$ and $\Phi \in \mathcal{S}(\mathcal{L})$. Then Φ is unique up to a scale factor in its argument.*

Source: Theorem 1.5 of Biggins and Kyprianou (1997). □

The main new feature of this result was that it gave uniqueness when the solution to (1) in $\mathcal{L}$ had an infinite mean. Here the following result, which includes Proposition 2 as a special case, will be proved.

Theorem 1 *Assume that (A) holds and $\Phi \in \mathcal{S}(\mathcal{L})$. Then Φ is unique up to a scale factor in its argument.*

This improves on Proposition 2 in two ways. It dispenses with the requirement that $v(\gamma) < \infty$ for some $\gamma > 1$ and, more significantly, it includes also the case when $v'(1) = 0$.

It is worth pointing out, but without going into detail, that one continuous analogue of (1) is the equation giving a travelling wave solution of a particular speed for the partial differential equation know as the KPP equation; in that context, $v'(1) = 0$ gives the wave of smallest speed. This connection, which receives further comment in Section 5, illustrates that the case where $v'(1) = 0$ is likely to be both subtle and important. Clearly it marks the boundary of the cases covered by $v'(1) \leq 0$ and is 'the boundary case' of the title.

Parts of the account in Biggins and Kyprianou (1997), which concerns the cases where $v'(1) < 0$, with additional assumptions, is relevant here. Many of the proofs there apply more widely, either with no change or with simple modifications. The presentation here aims to make the discussion and the statements of results self-contained, but it will be necessary to consult Biggins and Kyprianou (1997) for details in some proofs.

The functional equation relates in a natural way to certain martingales in the branching random walk. This relationship and recent results for the martingales, obtained in Biggins and Kyprianou (2001a), lead to rather precise information on the behaviour of solutions to (1) near zero when $v'(1) = 0$. To describe this behaviour easily, let L be given by

$$L(\psi) = \frac{1 - \Phi(\psi)}{\psi}, \tag{2}$$

where $\Phi \in \mathcal{S}(\mathcal{L})$. Since $\Phi \in \mathcal{L}$, L is a Laplace transform of a measure on $(0, \infty)$ and hence is decreasing in ψ, and then $L(0+)$ is finite exactly when the random variable corresponding to Φ has a finite mean. Hence, the next theorem implies that in the boundary case every solution to (1) has an infinite mean. A new assumption occurs here:

(V) $$v''(1) = \int x^2 e^{-x} \mu(dx) < \infty.$$

Theorem 2 *Assume $v'(1) = 0$, (A) and (V) hold and $\Phi \in \mathcal{S}(\mathcal{L})$. Then $(-\log \psi)^{-1} L(\psi)$ has a limit as $\psi \downarrow 0$ and the limit is strictly positive but may be infinite.*

The assumption (A) implies that $\int_{-\infty}^{0} x^2 e^{-x} \mu(dx) < \infty$, which is 'half' of (V), and so, in the statement of the previous theorem and the next one, (V) could have been rephrased to reflect this. However, (V) is used in intermediate results where (A) is not imposed.

To say more we need to introduce the following non-negative random variables:

$$G_1 = -\sum A_i \log A_i I(A_i < 1) \text{ and } \Gamma^{(s)} = \sum A_i I(A_i > e^{-s}).$$

Note that $\Gamma^{(s)} \uparrow \Gamma^{(\infty)} = \sum A_i$ as $s \uparrow \infty$. Also, let $\phi(x) = \log\log\log x$, $L_1(x) = (\log x)\phi(x)$, $L_2(x) = (\log x)^2\phi(x)$, $L_3(x) = (\log x)/\phi(x)$ and $L_4(x) = (\log x)^2/\phi(x)$; the key point about these functions is that L_1 and L_3 are similar to each other and to $\log x$ and L_2 and L_4 are similar to each other and to $(\log x)^2$. Hence the moment conditions in parts (a) and (b) of the next theorem are close to each other, but they do not form a dichotomy, there are cases in between.

Theorem 3 *Assume $v'(1) = 0$, (A) and (V) hold and $\Phi \in \mathcal{S}(\mathcal{L})$.*
(a) If both

$$E[G_1 L_1(G_1)] < \infty \text{ and } E[\Gamma^{(\infty)} L_2(\Gamma^{(\infty)})] < \infty$$

then

$$\lim_{\psi \downarrow 0} (-\log \psi)^{-1} L(\psi) \in (0, \infty).$$

(b) If

$$E\left[G_1 L_3(G_1)\right] = \infty \quad \text{or} \quad E\left[\Gamma^{(s)} L_4(\Gamma^{(s)})\right] = \infty \text{ for some } s$$

then

$$\lim_{\psi \downarrow 0} (-\log \psi)^{-1} L(\psi) = \infty.$$

This result improves the known results about the functional equation for this case contained in Theorem 1.4 of Liu (1998).

We finish this introduction with an overview of the rest of the paper. The aforementioned relationship with the branching random walk and some martingales arising from the functional equation are described in the next section. These allow the functional equation (1) to be transformed to another of the same form but satisfying stronger assumptions. This reduction is described in Section 3 and used to prove Theorem 1 from Proposition 2. Section 4 illustrates further the usefulness of this reduction and prepares the ground for the proof of Theorems 2 and 3, which are given in the final two sections.

2 Multiplicative martingales

There is a natural (one to one) correspondence, already hinted at, between the framework introduced and the branching random walk, a connection that is the key to some of the proofs. Specifically, let the point process Z (with points at $\{-\log A_i : A_i > 0\}$) be used to define a branching random walk in the usual way, with independent copies of Z being used to give the positions of each family relative to its parent's position. Ignoring positions gives a Galton-Watson process with (almost surely finite) family size N. People are labelled by their ancestry (the Ulam-Harris labelling) and the generation of u is $|u|$. Let z_u be the position of u, so that $\{z_u : |u| = 1\}$ is a copy of $\{-\log A_i : A_i > 0\}$. Then the assumption (H) translates to

(H) $$\int \mu(dx) > 1, \int e^{-x}\mu(dx) = 1 \text{ and } \int x e^{-x}\mu(dx) \geq 0.$$

Let $B_n = \inf\{z_u : |u| = n\}$, the position of the left-most person in the nth generation, which is taken to be infinite when the branching process has already died out by then.

Proposition 3 *The assumption (H) is enough to ensure that $B_n \to \infty$ almost surely.*

Source: Theorem 3 of Biggins (1998) or Lemma 7.2 in Liu (1998). □

The first result is easy to establish. It is natural to call the martingales it describes multiplicative martingales.

Proposition 4 *Let $\Phi \in \mathcal{S}(\mathcal{L})$. Then, for each $\psi > 0$*

$$\prod_{|u|=n} \Phi\left(\psi e^{-z_u}\right) \quad \textit{for } n = 0, 1, 2, 3, \ldots,$$

is a bounded martingale, which converges in mean and almost surely to $M(\psi)$. Furthermore, $EM(\psi) = \Phi(\psi)$, and so $P(M(\psi) < 1) > 0$ for all $\psi > 0$.

Source: Theorem 3.1 and Corollary 3.2 in Biggins and Kyprianou (1997).

In random walk theory the ladder height, the first point in $(0, \infty)$ reached by the random walk, is an important concept. From a random walk's trajectory, a sequence of successive independent identically distributed ladder heights can be constructed, each new one arising as the first overshoot of the previous maximum. Analogous ideas are important here.

For the branching random walk corresponding to A let

$$\mathcal{C} = \{u : z_u > 0 \text{ but } z_v \leq 0 \text{ for } v < u\}, \tag{3}$$

where $v < u$ means v is an ancestor of u. Hence $\mathcal{C}$ identifies the individuals who are the first in their lines of descent to be to the right of 0. The collection $\{z_u : u \in \mathcal{C}\}$ has the same role here as the first ladder height in a random walk. This motivates the next construction.

Starting from the initial ancestor, follow a line of descent down to its first member to the right of 0; doing this for all lines of descent produces $\mathcal{C}$. Regard the members of $\mathcal{C}$ as the children of the initial ancestor, rather than simply descendants; the resulting point process of children's positions, $\{z_u : u \in \mathcal{C}\}$, is concentrated on $(0, \infty)$ by arrangement. Now, pick a member of $\mathcal{C}$; follow a line of descent from this individual down to the first member to the right of the member picked; doing this for all lines of descent from the member picked produces a copy of $\{z_u : u \in \mathcal{C}\}$. This can be done for each member of $\mathcal{C}$ to produce a family for each of them, giving a 'second generation'. In the same way, families can be identified for these 'second generation' individuals and so on. The positions of individuals in this embedded process, which are all in $(0, \infty)$, can now be interpreted as birth times. The result is a general branching process, also called a Crump-Mode-Jagers (CMJ) process, constructed from individuals and their positions in the branching random walk.

It is possible to describe explicitly which individuals occur in the embedded process. In the branching random walk, for $t \geq 0$ let

$$\mathcal{C}(t) = \{u : z_u > t \text{ but } z_v \leq t \text{ for } v < u\},$$

so that $\mathcal{C}(0) = \mathcal{C}$, and let $\mathcal{C}(t)$ be the initial ancestor for $t < 0$. The individuals in the branching random walk that occur in the CMJ process are exactly those in $\mathcal{C}(t)$ as t

varies. The variable t can be interpreted as time. Then $\mathcal{C}(t)$ is what is called the coming generation for the CMJ process; it consists of the individuals born after t whose mothers are born no later than t. This whole construction is discussed more formally in Section 8 of Biggins and Kyprianou (1997); the relevant aspect is summarised in the next result. Naturally, the process constructed is called the embedded CMJ process.

Proposition 5 *The individuals $\{u : u \in \mathcal{C}(t)$ for some $t\}$, with the mother of u defined to be u's closest ancestor in the collection and u's birth time being z_u, form a general (CMJ) branching process with reproduction point process $\{z_u : u \in \mathcal{C}\}$.*

The final assertion of the next Proposition is a functional equation of the form (1), but with a different A. This transformation of the problem is examined further in the next section. In order to make it clear why that final assertion holds, multiplicative martingales like those defined in Proposition 4, but with the products taken over $\mathcal{C}(t)$, are introduced. These martingales play further part in the development here.

Proposition 6 *Let $\Phi \in \mathcal{S}(\mathcal{L})$. For $-\infty < t < \infty$, let*

$$M_t(\psi) = \prod_{u \in \mathcal{C}(t)} \Phi(\psi e^{-z_u}).$$

For each $\psi \geq 0$, $M_t(\psi)$ is a bounded martingale. In particular, Φ satisfies

$$\Phi(\psi) = E\left[\prod_{u \in \mathcal{C}} \Phi(\psi e^{-z_u})\right].$$

Source: Theorem 6.2 and Lemma 8.1 in Biggins and Kyprianou (1997). □

3 Reduction of the functional equation.

A major element in the approach here is the reduction of certain cases to simpler ones with stronger assumptions; this reduction is made precise in the next result. Given A, let A^* be the numbers $\{e^{-z_u} : u \in \mathcal{C}\}$, defined by (3), in decreasing order. Objects derived from A, like N and μ, have counterparts for A^*, denoted by N^*, μ^* and so on. The reproduction point process of the embedded CMJ process, introduced in the previous section, is $\{z_u : u \in \mathcal{C}\}$, which has intensity measure μ^*. Hence the next theorem can easily be reinterpreted to give properties of μ^*.

Theorem 4 *Let $\Phi \in \mathcal{S}(\mathcal{L})$.*
(a) Then Φ is also a solution to

$$\Phi(\psi) = E\left[\prod \Phi(\psi A_i^*)\right]$$

and $\max A_i^* < 1$.
(b) If (H) holds for A then it also holds for A^*.
(c) If (A) holds for A then it also holds for A^* with the same θ.
(d) If $P(N < \infty) = 1$ then $P(N^* < \infty) = 1$.

It is worth stressing that not all properties transfer exactly; for example, (V) for A does not imply (V) for A^*.

Before giving the proof, we need the following result, linking quantities of interest to expectations for random walk.

Proposition 7 *Let $S_0 = 0$ and let S_n be the sum of n independent identically distributed variables with law $e^{-x}\mu(dx)$. Then*

$$E\sum_{|u|=n} e^{-z_u} f(z_v : v \leq u) = E(f(S_k : k \leq n))$$

for all (measurable) functions f. In particular, taking $n = 1$ and writing in terms of A,

$$E\sum A_i f(-\log A_i) = E(f(S_1)).$$

Source: Lemma 4.1(iii) of Biggins and Kyprianou (1997); see also p289 of Durrett and Liggett (1983) as well as Lemma 1 of Bingham and Doney (1975). □

Notice that, by (H), $ES_1 = -v'(1) \geq 0$. Hence the random walk $S = \{S_n : n \geq 0\}$ has a non-negative drift. Let $\tau = \inf\{n \geq 0 : S_n \in (0, \infty)\}$, which must be finite almost surely because $ES_1 \geq 0$. Then S_τ is the first strict increasing ladder height of S.

Proof of Theorem 4. Part (a) is just a restatement of the final part of Proposition 6 and the fact that by definition all terms in $\{z_u : u \in \mathcal{C}\}$ are strictly positive.

Since

$$\exp(v^*(\theta)) = E\left[\sum (A_i^*)^\theta\right] = E\sum_{u\in\mathcal{C}} e^{-\theta z_u},$$

to prove (b), that is that (H) holds for v^*, we must show that

$$E|\mathcal{C}| > 1, \quad E\sum_{u\in\mathcal{C}} e^{-z_u} = 1 \text{ and } E\sum_{u\in\mathcal{C}} z_u e^{-z_u} \geq 0.$$

The last of these is immediate from the positivity of $\{z_u : u \in \mathcal{C}\}$. For the second note, using Proposition 7, that

$$\begin{aligned} E\sum_{u\in\mathcal{C}} f(z_u)e^{-z_u} &= \sum_{n\geq 1} E\sum_{|u|=n} I\left(z_u > 0, z_v \leq 0 \text{ for all } v < u\right) f(z_u)e^{-z_u} \\ &= \sum_{n\geq 1} E\left[I\left(S_n > 0, S_k \leq 0 \text{ for all } k < n\right) f(S_n)\right] \\ &= E\left[I(S_\tau < \infty) f(S_\tau)\right]. \end{aligned}$$

In particular,

$$E\sum_{u\in\mathcal{C}} e^{-z_u} = P\left(\tau < \infty\right) = 1$$

and then

$$E|\mathcal{C}| > E\sum_{u\in\mathcal{C}} e^{-z_u} = 1.$$

For (c) note that

$$E\left[\sum_{u\in\mathcal{C}} e^{-\theta z_u}\right] = E\left[\sum_{u\in\mathcal{C}} e^{(1-\theta)z_u} e^{-z_u}\right] = Ee^{(1-\theta)S_\tau}.$$

Thus the required finiteness reduces to the ladder height S_τ having a suitable exponential tail. Now

$$\infty > e^{v(\theta)} = \int e^{-\theta x}\mu(dx) = \int e^{(1-\theta)x}e^{-x}\mu(dx) = E[e^{(1-\theta)S_1}]$$

and so the tails of the increment distribution of the random walk decay exponentially. This implies, by standard random walk theory, in particular, XII(3.6a) in Feller (1971), that the same is true of S_τ.

Since $B_n \to \infty$, $\mathcal{C}$ is contained entirely within some finite number of generations. Since N, the family size, is finite this forces $|\mathcal{C}|$ to be finite, giving (d). □

Proof of Theorem 1. When $\max A_i < 1$ it is clear that $v(\gamma) < \infty$ for some $\gamma > 1$ and that $v'(1) < 0$. Hence, Theorem 4 reduces the cases being considered in Theorem 1 to those covered by Proposition 2. □

4 Slow variation and its consequences

To prove Theorems 2 and 3 a little more information about the behaviour of the multiplicative martingales is needed.

Proposition 8 *Let $\Phi \in \mathcal{S}(\mathcal{L})$. Then $L(\psi)$ is slowly varying as $\psi \downarrow 0$.*

Source: Theorem 1.4 of Biggins and Kyprianou (1997) when $v'(1) < 0$; Theorem 2 of Kyprianou (1998) when $v'(1) = 0$.

However, it is worth noting that Theorem 4 transforms cases where $v'(1) = 0$ into ones where $v'(1) < 0$ and then Theorem 1.4 of Biggins and Kyprianou (1997) applies. In this way the use of Theorem 2 of Kyprianou (1998) could be circumvented. □

Arguably, the next result should be a Proposition, since it is a routine extension of what is already known.

Lemma 1 *Let $\Phi \in \mathcal{S}(\mathcal{L})$, L be given by (2) and $M(\psi)$ be the limit introduced in Proposition 4.*
(a)

$$\lim_{n\to\infty} \sum_{|u|=n} e^{-z_u} L(e^{-z_u}) = -\log M(1).$$

(b) $M(\psi) = M(1)^\psi$ and so $\Phi(\psi) = E[e^{\log M(1)\psi}]$.

Proof. These results are proved in Lemmas 5.1 and 5.2 of Biggins and Kyprianou (1997). Those proofs work here, but now use Propositions 3 and 8 for the facts that $B_n \to \infty$ and L is slowly varying. □

Proposition 9 *Assume $v'(1) < 0$, (A) holds, $v(\gamma) < \infty$ for some $\gamma > 1$ and $\Phi \in \mathcal{S}(\mathcal{L})$. Then*

$$\lim_{t\to\infty} L(e^{-t}) \sum_{u\in\mathcal{C}(t)} e^{-z_u} = W, \tag{4}$$

where W has Laplace transform Φ.

Source: Theorem 8.6 in Biggins and Kyprianou (1997). □

Theorem 5 *Assume (A) holds and $\Phi \in \mathcal{S}(\mathcal{L})$. Then the conclusion of Proposition 9 holds.*

Proof. Proposition 5 describes a CMJ embedded in the original branching random walk. Theorem 4 shows that the embedded CMJ, viewed as a branching random walk with only positive steps, satisfies all the conditions of Proposition 9. The conclusion is then that (4) holds for the embedded CMJ process of this branching random walk with only positive steps, but in such a case the embedded process is identical to the original one. Hence Proposition 9 does indeed produce the result. □

Further argument, of the kind in Biggins and Kyprianou (1997), shows that $W = -\log M(1)$, but this connection is not needed for the subsequent arguments.

5 The derivative martingale

In this section will consider only the case where $v'(0) = \int xe^{-x}\mu(dx) = 0$, that is the boundary case. We will describe some properties of a martingale that is intimately related to the properties of the functional equation in this case.

Let

$$\partial W_n = \sum_{|u|=n} z_u e^{-z_u};$$

then it is straightforward to check that ∂W_n is a martingale. It is called the derivative martingale because its form can be derived by differentiating $\sum_{|u|=n} e^{-\theta z_u - nv(\theta)}$, which is also a martingale, with respect to θ and then setting θ to one. The martingale ∂W_n has been considered in Kyprianou (1998) and Liu (2000) and its analogue for branching Brownian motion has been discussed by several authors — Neveu (1988) and Harris (1999), for example. In the branching Brownian motion context, it is the travelling wave solutions to the KPP equation that provide the analogue of solutions to the functional equation. Classical theory of ordinary differential equations provides existence, uniqueness and aspects of the asymptotic behaviour of these travelling waves; hence, these properties form part of the starting point in Neveu's study and earlier ones. In contrast, Harris (1999) seeks properties of the solutions through arguments based on associated martingales, which is the approach taken here.

The derivative martingale is one of the main examples in Biggins and Kyprianou (2001a), where general results on martingale convergence in branching processes are discussed.

Proposition 10 *When $v'(1) = 0$ and (V) holds, the martingale ∂W_n converges to a finite non-negative limit, Δ, almost surely. Then*

$$\Delta = \sum_{|u|=1} e^{-z_u}\Delta_u,$$

where, given the first generation, for each u such that $|u| = 1$, Δ_u are independent copies of Δ. Furthermore, $P(\Delta = 0)$ is either equal to the extinction probability or equal to one.

Source: Theorem 4.1 in Biggins and Kyprianou (2001a). □

This result shows that the transform of Δ satisfies (1) and will have a transform in $\mathcal{L}$ when Δ is not identically zero. Whether the martingale limit Δ is zero or not is related to the behaviour of the solution to (1) near the origin. The precise relationship is formulated in the next theorem, the proof of which is deferred to Section 6.

Theorem 6 *Suppose $v'(1) = 0$, (A) and (V) hold and $\Phi \in \mathcal{S}(\mathcal{L})$. Then $P(\Delta > 0) > 0$ if and only if*

$$\lim_{\psi \downarrow 0} \left((-\log \psi)^{-1} L(\psi) \right) = c \in (0, \infty); \tag{5}$$

furthermore $P(\Delta = 0) = 1$ if and only if $(-\log \psi)^{-1} L(\psi) \to \infty$ as $\psi \downarrow 0$. In fact, (A) is not needed for the 'if' parts here.

Proof of Theorem 2. This result is contained in Theorem 6. □

Information on when Δ is not zero, and when it is, is given in the next result, with the notation used in Theorem 3.

Proposition 11 *Assume $v'(1) = 0$ and that (V) holds.*
(a) If both $E[G_1 L_1(G_1)] < \infty$ and $E[\Gamma^{(\infty)} L_2(\Gamma^{(\infty)})] < \infty$ then Δ is not identically zero.
(b) If $E\left[G_1 L_3(G_1)\right] = \infty$ or $E\left[\Gamma^{(s)} L_4(\Gamma^{(s)})\right] = \infty$ for some $s > 0$ then $\Delta = 0$ almost surely.

Source: Theorem 4.1 of Biggins and Kyprianou (2001a). □

Proof of Theorem 3. Combine Theorem 6 and Proposition 11. □

Some results on the relationship between the limiting behaviour in (5), the limit Δ, and the uniqueness of the solution to (1), have been obtained previously, in Kyprianou (1998) and Liu (2000) under moment conditions; those studies approach the convergence of ∂W_n and uniqueness through (5). Proposition 11 shows that that the asymptotic (5) does not always hold, limiting that approach to uniqueness.

6 Proof of Theorem 6

The preliminary lemma borrows heavily from the proof of Theorem 8.6 in Biggins and Kyprianou (1997).

Lemma 2 *Suppose $v'(1) = 0$, and that (A) and (V) hold. Then*

$$t \sum_{u \in \mathcal{C}(t)} e^{-z_u} \to \Delta$$

almost surely, as $t \to \infty$.

Proof. Note first that, for $x \geq 1$ and any $\epsilon \in (0, 1)$, $\epsilon x \leq e^{\epsilon(x-1)}$ and thus, for $u \in \mathcal{C}(t)$ and $t \geq 1$,

$$z_u/t \leq \epsilon^{-1} e^{\epsilon(z_u - t)}.$$

Take $(1-\epsilon) \geq \theta$, where θ comes from (A). A routine application of Theorem 6.3 in Nerman (1981), following closely the corresponding calculation in the proof of Theorem 8.6 in Biggins and Kyprianou (1997), shows that

$$\lim_{c\to\infty} \lim_{t\to\infty} \frac{\sum_{u\in\mathcal{C}(t)} e^{-(1-\epsilon)(z_u-t)} I(z_u > t+c)}{\sum_{u\in\mathcal{C}(t)} e^{-(z_u-t)}} = 0 \quad \text{a.s.},$$

when $\mathcal{C}(t)$ is not eventually empty.

Now, for $t \geq 1$, small $\epsilon > 0$ and $\mathcal{C}(t)$ non-empty,

$$\begin{aligned} 1 \leq \frac{\sum_{u\in\mathcal{C}(t)} z_u e^{-z_u}}{\sum_{u\in\mathcal{C}(t)} te^{-z_u}} &= \frac{\sum_{u\in\mathcal{C}(t)} z_u e^{-(z_u-t)}}{\sum_{u\in\mathcal{C}(t)} te^{-(z_u-t)}} \\ &\leq \frac{t+c}{t} + \frac{\sum_{u\in\mathcal{C}(t)} (z_u/t) e^{-(z_u-t)} I(z_u > t+c)}{\sum_{u\in\mathcal{C}(t)} e^{-(z_u-t)}} \\ &\leq \frac{t+c}{t} + \epsilon^{-1} \frac{\sum_{u\in\mathcal{C}(t)} e^{-(1-\epsilon)(z_u-t)} I(z_u > t+c)}{\sum_{u\in\mathcal{C}(t)} e^{-(z_u-t)}} \end{aligned}$$

which tends to one as t and then c tend to infinity. The proof is completed by noting that

$$\sum_{u\in\mathcal{C}(t)} z_u e^{-z_u} \to \Delta$$

almost surely as $t \to \infty$, by Theorem 4.2 in Biggins and Kyprianou (2001a). □

Proof of Theorem 6. Suppose that $((-\log\psi)^{-1}L(\psi))$ has a limit ℓ as $\psi \downarrow 0$. Then, using Lemma 1 and Proposition 3

$$-\log M(1) \;=\; \lim_{n\to\infty} \sum_{|u|=n} e^{-z_u} L(e^{-z_u}) \sim \ell \lim_{n\to\infty} \sum_{|u|=n} z_u e^{-z_u} = \ell\Delta,$$

and $-\log M(1)$ is finite and not identically zero. This proves the 'if' parts of the result. This part of the argument is based on the proof of Theorem 2.5 of Liu (2000); see also Theorem 3 of Kyprianou (1998).

To go the other way, let Δ be the limit of ∂W_n and let $\Phi \in \mathcal{S}(\mathcal{L})$. Then, by Theorem 5 and Lemma 2

$$\frac{\Delta}{W} = \lim_{t\uparrow\infty} \frac{t\sum_{u\in\mathcal{C}(t)} e^{-z_u}}{L(e^{-t})\sum_{u\in\mathcal{C}(t)} e^{-z_u}} = \lim_{t\uparrow\infty} \frac{t}{L(e^{-t})} = \lim_{t\uparrow\infty} \frac{te^{-t}}{1-\Phi(e^{-t})},$$

which must be a (non-random) constant. The constant is only zero when Δ is identically zero; otherwise, (5) holds. □

The first half of the proof just given is unnecessary when (A) holds, since the second half actually gives the claimed equivalence. Hence this treatment could have omitted Proposition 8 and Lemma 1 by sacrificing the last assertion in Thoerem 6.

The idea that the convergence described in Lemma 2 produces information on the asymptotics of the functional equation occurs, in the branching Brownian motion context with non-trivial Δ, in Kyprianou (2003). It is also worth noting that Lemma 2 provides a

Seneta-Heyde norming for the Nerman martingale $\sum_{u\in\mathcal{C}(t)} e^{-z_u}$ associated with the particular CMJ process arising here. The existence of such a norming in general is covered by Theorem 7.2 of Biggins and Kyprianou (1997). The special structure here means that the slowly varying function in the general theorem is the logarithm.

Lyons (1997) shows that when $v'(1) = 0$ the non-negative martingale $W_n = \sum_{|u|=n} e^{-z_u}$ converges almost surely to zero. In the same spirit as Theorem 1.2 in Biggins and Kyprianou (1997), it is natural to wonder whether there are constants c_n such that W_n/c_n converges. In Biggins and Kyprianou (1997), the approach to this question, which we have not been able to settle in the present context, needs a 'law of large numbers' which would say, roughly, $W_{n+1}/W_n \to 1$ in probability.

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